The Feynman's Propagators for Non-Linear Schrödinger's Equations:[1]

José Maria Filardo Bassalo

Professor Aposentado da Universidade Federal do Pará

jmfbassalo@gmail.com

Daniel Gemaque da Silva

Professor da Escola Munguba do Jari, Vióría do Jari, AP

danicofisicaquantica@yahoo.com.br

Antonio Boulhosa Nassar

Extension Program-Department of Sciences, University of California, Los Angeles, California 90024

nassar@ucla.edu

Mauro Sérgio Dorsa Cattani

Instituto de Física da USP, 05389-970, São Paulo, SP

mcattani@if.usp.br

Abstract: In this paper we study the Feynman propagators for eight non-linear Schrödinger equations, linearized along a classical trajectory, by using the quantum mechanical formalism of the de Broglie-Bohm.

PACS 03.65 - Quantum Mechanics

1. Introduction: The Feynman Propagator

In 1948,[1] R. P. Feynman formulated the following principle of minimum action for the quantum mechanics:

The transition amplitude between the states | a > and | b > of a quantum-mechanical system is given by the sum of the elementary contributions, one for each trajectory passing by | a > at the time t_a and by | b > at the time t_b. Each one of these contributions have the same modulus, but its phase is the classical action $S_{c\ell}$ for each trajectory.

[1] This article it is dedicated to the Memory of the Paulo de Tarso Santos Alencar (1940-2011), our colaborator in some of the chapters of that text.

This principle is represented by the following expression known as the "Feynman propagator":

$$K(b, a) = \int_a^b exp\,[\frac{i}{\hbar} S(b, a)]\, D\, x(t)\ , \quad (1.1)$$

where $S(b, a)$ is the *classical action* given by:

$$S(b, a) = \int_{t_a}^{t_b} L\,(x, \dot{x}, t)\, dt\ , \quad (1.2)$$

$L(x, \dot{x}, t)$ is the Lagrangean and $D\, x(t)$ is the Feynman's Measurement. It indicates that we must perform the integration taking into account all the ways connecting the states $|\, a >$ and $|\, b >$.

The eq. (1.1) which defines $K(b, a)$ is called *path integral* or *Feynman integral* and the Schrödinger wavefunction $\Psi(x, t)$ of any physical system is given by (we indicate the initial position and initial time by x_o and t_o, respectively): [2]

$$\Psi(x, t) = \int_{-\infty}^{+\infty} K(x, x_o; t, t_o)\, \Psi(x_o, t_o)\, dx_o\ , \quad (1.3)$$

with the *quantum causality condition*: [3]

$$\lim_{t,\, t_o \to 0} K(x, x_o; t, t_o) = \delta(x - x_o)\ . \quad (1.4)$$

2. The Feynman-de Broglie-Bohm Propagator for the Non-Linear Schrödinger Equations

Now, let us calculated the Feynman propagators for eight non-linear Schrödinger equations, linearized along a classical trajectory, by using the quantum mechanical formalism of the de Broglie-Bohm. [4]

2.1. The Bialynicki-Birula-Mycielski Equation

In 1976 and 1979, [5] I. Bialynicki-Birula and J. Mycielski proposed a non-linear Schrödinger equation, to represent time dependent physical systems, defined by:

$$i\,\hbar\, \frac{\partial \Psi(x, t)}{\partial t} = -\, \frac{\hbar^2}{2\, m}\, \frac{\partial^2\, \Psi(x, t)}{\partial x^2} +$$

$$+ \{V(x, t) - \frac{\hbar\, \lambda}{2}\, \ell n\, [\Psi(x, t)\, \Psi^*(x, t)]\} \times \Psi(x, t)\ , \quad (2.1.1)$$

where $\Psi(x, t)$ and $V(x, t)$ are, respectively, the wave function and the time dependent potential of the physical system in study, λ is a constant, and (*) means complex conjugate.

2.1.1. The Wave Function of the Bialynicki-Birula-Mycielski Equation

Writting the wave function $\Psi(x, t)$ in the polar form defined by the Madelung-Bohm transformation [6,7] we obtain:

$$\Psi(x, t) = \varphi(x, t) \times exp\,[i\,S(x, t)]\ , \qquad (2.1.1.1)$$

where $\varphi(x, t)$ will be defined in what follows.

Calculating the derivatives, temporal and spatial, of (2.1.1.1), we get [remembering that $exp\,[i\,S]$ is common factor]: [4]

$$\frac{\partial \Psi}{\partial t} = exp\,(i\,S)\,(\,\frac{\partial \varphi}{\partial t} + i\,\varphi\,\frac{\partial S}{\partial t}\,)\ , \qquad (2.1.1.2a)$$

$$\frac{\partial^2 \Psi}{\partial x^2} = exp\,(i\,S)\,[\,\frac{\partial^2 \varphi}{\partial x^2} + 2\,i\,\frac{\partial S}{\partial x}\,\frac{\partial \varphi}{\partial x} + i\,\varphi\,\frac{\partial^2 S}{\partial x^2} - \varphi\,(\,\frac{\partial S}{\partial x}\,)^2\,]\ , \qquad (2.1.1.2b)$$

$$i\,\hbar\,(\,\frac{\partial \varphi}{\partial t} + i\,\varphi\,\frac{\partial S}{\partial t}\,) = -\,\frac{\hbar^2}{2\,m}\,[\,\frac{\partial^2 \varphi}{\partial x^2} +$$

$$+\,2\,i\,\frac{\partial S}{\partial x}\,\frac{\partial \varphi}{\partial x} + i\,\varphi\,\frac{\partial^2 S}{\partial x^2} - \varphi\,(\,\frac{\partial S}{\partial x}\,)^2\,] + [\,V(x, t) - \frac{\hbar\,\lambda}{2}\,\ell n\,(\varphi)^2\,]\,\varphi\ , \qquad (2.1.1.3)$$

Separating the real and imaginary parts of the relation (2.1.1.3), results:

a) <u>imaginary part</u>

$$\frac{\hbar}{\varphi}\,\frac{\partial \varphi}{\partial t} = -\,\frac{\hbar^2}{2\,m}\,(\,2\,\frac{1}{\varphi}\,\frac{\partial S}{\partial x}\,\frac{\partial \varphi}{\partial x} + \frac{\partial^2 S}{\partial x^2}\,)\ , \qquad (2.1.1.4)$$

b) <u>real part</u>

$$-\,\hbar\,\frac{\partial S}{\partial t} = -\,\frac{\hbar^2}{2\,m}\,[\,\frac{1}{\varphi}\,\frac{\partial^2 \varphi}{\partial x^2} - (\,\frac{\partial S}{\partial x}\,)^2\,] + V(x, t) - \frac{\hbar\,\lambda}{2}\,\ell n\,(\varphi)^2\ . \qquad (2.1.1.5)$$

2.1.2. Dynamics of the Bialynicki-Birula-Mycielski Equation

Now, let us see the correlation between the expressions (2.1.1.4-5) and the traditional equations of the Ideal Fluid Dynamics [8] a) *Continuity Equation*, b) *Euler's equation*. To do this let us perform the following correspondences:

<u>Quantum density probability</u>: $\quad |\,\Psi(x, t)\,|^2 = \Psi^*(x, t)\,\Psi(x, t) \quad \leftrightarrow$

<u>Quantum mass density</u>: $\quad \varrho(x, t) = \varphi^2(x, t) \ \leftrightarrow\ \sqrt{\varrho} = \varphi\ , \qquad (2.1.2.1a,b)$

$$\underline{\text{Gradient of the wave function}}: \quad \frac{\hbar}{m}\frac{\partial S(x,t)}{\partial x} \quad \longleftrightarrow$$

$$\underline{\text{Quantum velocity}}: \quad v_{qu}(x,t) \equiv v_{qu}, \quad (2.1.2.1c,d)$$

Bohm quantum potential:

$$V_{qu}(x,t) \equiv V_{qu} = -\left(\frac{\hbar^2}{2\,m\,\varphi}\right)\frac{\partial^2\varphi}{\partial x^2} = -\frac{\hbar^2}{2\,m}\frac{1}{\sqrt{\varrho}}\frac{\partial^2\sqrt{\varrho}}{\partial x^2}, \quad (2.1.2.1e,f)$$

Putting the relations (2.1.2.1a-d) into the equation (2.1.1.4) and considering that $\partial(\ell n\, x)/\partial y = (1/x)\,(\partial x/\partial y)$ and $\ell n(x^m) = m\,\ell n\, x$, we get: [4]

$$\frac{\partial}{\partial t}[\ell n\,(\varphi^2)] = -\frac{\hbar}{m}\left\{\frac{\partial^2 S}{\partial x^2} + \frac{\partial S}{\partial s}\frac{\partial}{\partial x}[\ell n\,(\varphi^2)]\right\} \;\rightarrow$$

$$\frac{\partial}{\partial t}(\ell n\,\varrho) = -\frac{\hbar}{m}\left[\frac{\partial^2 S}{\partial x^2} + \frac{\partial S}{\partial x}\frac{\partial}{\partial x}(\ell n\,\varrho)\right] = -\frac{\hbar}{m}\left[\frac{\partial^2 S}{\partial x^2} + \frac{\partial S}{\partial x}\frac{1}{\varrho}\frac{\partial\varrho}{\partial x}\right] =$$

$$= -\frac{\partial}{\partial x}\left(\frac{\hbar}{m}\frac{\partial S}{\partial x}\right) - \left(\frac{\hbar}{m}\frac{\partial S}{\partial x}\right)\frac{1}{\varrho}\frac{\partial\varrho}{\partial x} \;\rightarrow\; \frac{1}{\varrho}\frac{\partial\varrho}{\partial t} + \frac{\partial v_{qu}}{\partial x} + \frac{v_{qu}}{\varrho}\frac{\partial\varrho}{\partial x} = 0 \;\rightarrow$$

$$\frac{\partial\varrho}{\partial x} + \frac{\partial(\varrho\, v_{qu})}{\partial x} = 0, \quad (2.1.2.2)$$

which represents the *Continuity Equation* or *Mass Conservation Law* of the Fluid Dynamics. We must note that this expression also shows a <u>coherent effect</u> in the physical system represented by the Bialynicki-Birula-Mycielski Equation (*BBM−E*) [eq. (2.1.1)].

Now, let us obtained another dynamic equation of the *BB−ME*. So, differentiating the eq. relation (2.1.1.5) with respect x and using the eqs. (2.1.2.1a-e), we obtain:

$$-\hbar\frac{\partial^2 S}{\partial x\,\partial t} = -\frac{\hbar^2}{2m}\frac{\partial}{\partial x}\left[\frac{1}{\varphi}\frac{\partial^2\varphi}{\partial x^2} - \left(\frac{\partial S}{\partial x}\right)^2\right] + \frac{\partial}{\partial x}\left[V(x,t) - \frac{\hbar\,\lambda}{2}\,\ell n\,(\varphi^2)\right] \;\rightarrow$$

$$\frac{\partial}{\partial t}\left(\frac{\hbar}{m}\frac{\partial S}{\partial x}\right) =$$

Erro!

$$= \frac{\partial}{\partial x}\left[\frac{\hbar^2}{2\,m^2}\frac{1}{\varphi}\frac{\partial^2\varphi}{\partial x^2} - \frac{V(x,t)}{m} + \frac{\hbar\,\lambda}{2\,m}\,\ell n\,(\varphi^2)\right] - \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\hbar}{m}\frac{\partial S}{\partial x}\right)^2 \;\rightarrow$$

$$\frac{\partial v_{qu}}{\partial t} + v_{qu} \frac{\partial v_{qu}}{\partial x} + \frac{1}{m} \frac{\partial}{\partial x} [\, V(x, t) + V_{qu}(x, t) - V_{BBM}(x, t)\,) = 0 \,, \qquad (2.1.2.3)$$

where:

$$V_{BBM}(x, t) = \frac{\hbar\,\lambda}{2} \,\ell n\, (\varphi^2) = \frac{\hbar\,\lambda}{2} \,\ell n\, \varrho \,, \qquad (2.1.2.4a,b)$$

is the *Bialynicki-Birula-Mycielski Potential*. We must observe that the eq. (2.1.2.3) is an equation similar to the *Euler Equation* which governs the motion of an ideal fluid.

Considering the *substantive differentiation* (local plus convective) or *hydrodynamic differentiation*: [8]

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v_{qu} \frac{\partial}{\partial x} \,, \qquad (2.1.2.5a)$$

and that:

$$v_{qu}(x, t)\, |_{x=x(t)} = \frac{dx}{dt} \,, \qquad (2.1.2.5b)$$

the eq. (2.1.2.3) could be written as:

$$m \frac{d^2x}{dt^2} = - \frac{\partial}{\partial x} [\, V(x, t) + V_{qu}(x, t) - V_{BBM}(x, t)\,] \;\; \rightarrow$$

$$m \frac{d^2x}{dt^2} = F_c(x, t)\, |_{x=x(t)} + F_{qu}(x, t)\, |_{x=x(t)} - F_{BBM}(x, t)\, |_{x=x(t)} \,. \qquad (2.1.2.6)$$

We note that the eq. (2.1.2.6) has a form of the *Second Newton Law*, being the first term of the second member is the *classical newtonian force*, the second is the *quantum bohmnian force*, and the three, is the *Bialynicki-Birula-Mycielski force*.

2.1.3 The Quantum Wave Packet of the Linearized Bialynicki-Birula-Mycielski Equation along a Classical Trajetory

In order to find the quantum wave packet of the linearized Bialynicki-Birula-Mycielski Equation (*BBM−E*) along a classical trajetory, let us the considerer the *ansatz*: [9]

$$\varrho(x, t) = [2\, \pi\, a^2(t)]^{-1/2} \times exp\, \{- \frac{[x - q(t)]^2}{2\, a^2(t)}\} \qquad (2.1.3.1a)$$

or [use eq. (2.1.2.1a,b)]:

$$\varphi(x, t) = [2\,\pi\, a^2(t)]^{-1/4} \times exp\,\{-\,\frac{[x - q(t)]^2}{4\, a^2(t)}\} \qquad (2.1.3.1b)$$

where $a(t)$ and $q(t) = <x>$ are auxiliary functions of time, to will be determined in what follows, representing the *width* and the *center of mass of wave packet*, respectively.

Differentiating the expression (2.1.3.1a) in the variable t, and remembering that x and t are independent variables, results:

$$\frac{\partial \varrho}{\partial t} = -\,\frac{1}{2}\,[2\,\pi\, a^2(t)]^{-3/2} \times [4\,\pi\, a(t)\, \dot{a}(t)] \times exp\,\{-\,\frac{[x - q(t)]^2}{2\, a^2(t)}\} +$$

$$+\,[2\,\pi\, a^2(t)]^{-1/2} \times exp\,\{-\,\frac{[x - q(t)]^2}{2\, a^2(t)}\} \times \frac{\partial}{\partial t}\{-\,\frac{[x - q(t)]^2}{2\, a^2(t)}\} =$$

$$-\,\varrho\,\{[2\,\pi\, \dot{a}(t)] \times [2\,\pi\, a^2(t)]^{-1} + \frac{4\, a^2(t)[x - q(t)]\times[-\dot{q}(t)] - 4a(t)\dot{a}(t)\times[x - q(t)]^2}{4a^4(t)}\} \rightarrow$$

$$\frac{\partial \varrho}{\partial t} = \varrho\,\{-\,\frac{\dot{a}(t)}{a(t)} + \frac{\dot{q}(t)}{a^2(t)}\,[x - q(t)] + \frac{\dot{a}(t)}{a^3(t)}\,[x - q(t)]^2\}\,. \qquad (2.1.3.2)$$

Substituting the eq. (2.1.3.2) into eq. (2.1.2.2) and integrating the result, we have (we consider null the integration constant):

$$\varrho\,\{-\,\frac{\dot{a}(t)}{a(t)} + \frac{\dot{q}(t)}{a^2(t)}\,[x - q(t)] + \frac{\dot{a}(t)}{a^3(t)}\,[x - q(t)]^2\} + \frac{\partial(\varrho\, v_{qu})}{\partial x} = 0 \quad \rightarrow$$

$$\int \frac{\partial(\varrho\, v_{qu})}{\partial x}\,\partial x = \int \varrho\,\{\frac{\dot{a}(t)}{a(t)} - \frac{\dot{q}(t)}{a^2(t)}\,[x - q(t)] - \frac{\dot{a}(t)}{a^3(t)}\,[x\, q(t)]^2\}\,\partial x \quad \rightarrow$$

$$\varrho\, v_{qu} = \int \varrho\,\{\frac{\dot{a}(t)}{a(t)} - \frac{[x - q(t)]}{a^2(t)} \times (\frac{\dot{a}(t)}{a(t)}\,[x - q(t)] + \dot{q}(t)\,)\}\,\partial x \quad \rightarrow$$

$$v_{qu} = \frac{1}{\varrho} \int \varrho\,\{\frac{\dot{a}(t)}{a(t)} - \frac{[x - q(t)]}{a^2(t)} \times (\frac{\dot{a}(t)}{a(t)}\,[x - q(t)] + \dot{q}(t)\,)\,\}\,\partial x\,. \qquad (2.1.3.3)$$

Now, using the eq. (2.1.3.1a), we can right that:

$$\frac{\partial}{\partial x}\,\{\varrho\,[\,\frac{\dot{a}(t)}{a(t)}\,[x - q(t)] + \dot{q}(t)\,]\} = \varrho\,\frac{\partial}{\partial x}\,[\,\frac{\dot{a}(t)}{a(t)}\,[x - q(t)] + \dot{q}(t)\,] + [\,\frac{\dot{a}(t)}{a(t)}\,[x - q(t)] + \dot{q}(t)\,]\,\frac{\partial \varrho}{\partial x} =$$

$$= \varrho \frac{\dot{a}(t)}{a(t)} + [\frac{\dot{a}(t)}{a(t)}[x - q(t)] + \dot{q}(t)] \times \frac{\partial}{\partial x}([2\pi a^2(t)]^{-1/2} \times exp\{-\frac{[x - q(t)]^2}{2 a^2(t)}\}) =$$

Erro!

$$= \varrho \frac{\dot{a}(t)}{a(t)} + [\frac{\dot{a}(t)}{a(t)}[x - q(t)] + \dot{q}(t)] \times \varrho \{-\frac{[x - q(t)]}{a^2(t)}\} \rightarrow$$

$$\frac{\partial}{\partial x}\{\varrho [\frac{\dot{a}(t)}{a(t)}[x - q(t)] + \dot{q}(t)]\} = \varrho\{\frac{\dot{a}}{a(t)} - \frac{[x - q(t)]}{a^2(t)} \times [\frac{\dot{a}(t)}{a(t)}[x - q(t)] + \dot{q}(t)]\} . \quad (2.1.3.4)$$

Substituting the eq. (2.1.3.4) into the eq. (2.1.3.3), results:

$$v_{qu} = \frac{1}{\varrho} \int \frac{\partial}{\partial x}\{\varrho [\frac{\dot{a}(t)}{a(t)}[x - q(t)] + \dot{q}(t)]\} \partial x \rightarrow$$

$$v_{qu}(x, t) \equiv \frac{dx(t)}{dt} = \frac{\dot{a}(t)}{a(t)}[x - q(t)] + \dot{q}(t). \quad (2.1.3.5)$$

We observe that the integration of the eq. (2.1.3.5) give us the *bohmnian quantum trajectory* of the physical system considered.

To obtain the quantum wave packet $[\Psi(x, t)]$ of the *BBM−E* given by eq. (2.1.1), let us expand the functions S(x, t), V(x t), V_{qu}(x, t) and V_{BBM}(x, t) around of $q(t)$ up to *second Taylor order*. [9] In this way, we have:

$$S(x, t) = S[q(t), t] + S'[q(t), t] \times [x - q(t)] + \frac{S''[q(t)\ t]}{2} \times [x - q(t)]^2, \quad (2.1.3.6)$$

$$V(x, t) = V[q(t), t] + V'[q(t), t] \times [x - q(t)] + \frac{V''[q(t), t]}{2} \times [x - q(t)]^2, \quad (2.1.3.7)$$

$$V_{qu}(x, t) = V_{qu}[q(t), t] + V'_{qu}[q(t), t] \times [x - q(t)] +$$

$$+ \frac{V''_{qu}[q(t), t]}{2} \times [x - q(t)]^2, \quad (2.1.3.8)$$

$$V_{BBM}(x, t) = V_{BBM}[q(t), t] + V'_{BBM}[q(t), t] \times [x - q(t)] +$$

$$+ \frac{V''_{BBM}[q(t), t]}{2} \times [x - q(t)]^2, \quad (2.1.3.9)$$

where (’) and " means, respectively, $\frac{\partial}{\partial q}$ and $\frac{\partial^2}{\partial q^2}$.

Differentiating the eq. (2.1.3.6) in the variable *x*, multiplying the result by ħ/*m*, using the eqs. (2.1.2.1c,d) and (2.1.3.5), results:

$$\frac{\hbar}{m}\frac{\partial S(x,t)}{\partial x} = \frac{\hbar}{m}\{S'[q(t),t] \times [x - q(t)] + \frac{S''[q(t)\ t]}{2} \times [x - q(t)]^2\} =$$

$$= v_{qu}(x,t) = \frac{\dot{a}(t)}{a(t)}[x - q(t)] + \dot{q}(t) \quad \rightarrow$$

$$S'[q(t),t] = \frac{m\,\dot{q}(t)}{\hbar}, \quad S''[q(t),t] = \frac{m\,\dot{a}(t)}{\hbar\, a(t)}. \qquad (2.1.3.10a,b)$$

Substituting the eq. (2.1.3.10a,b) into the eq. (2.1.3.6), we have:

$$S(x,t) = S_0(t) + \frac{m\,\dot{q}(t)}{\hbar}[x - q(t)] + \frac{m\,\dot{a}(t)}{2\,\hbar\, a(t)}[x - q(t)]^2, \qquad (2.1.3.11a)$$

where:

$$S_0(t) \equiv S[q(t), t], \qquad (2.1.3.11b)$$

are the *quantum action*.

Differentiating the eq. (2.1.3.11b) in relation to the time *t*, we obtain (remembering that $\partial x/\partial t = 0$):

$$\frac{\partial S}{\partial t} = \dot{S}_0(t) + \frac{\partial}{\partial t}\{\frac{m\,\dot{q}(t)}{\hbar}[x - q(t)]\} + \frac{\partial}{\partial t}\{\frac{m\,\dot{a}(t)}{2\,\hbar\, a(t)}[x - q(t)]^2\} \quad \rightarrow$$

$$\frac{\partial S}{\partial t} = \dot{S}_0(t)\ \frac{\ddot{q}(t)}{\hbar}[x - q(t)] - \frac{m\,\dot{q}^2(t)}{\hbar} +$$

$$+ \frac{m}{2\,\hbar}[\frac{\ddot{a}(t)}{a(t)} - \frac{\dot{a}^2(t)}{a^2(t)} \times [x - qt(t)]^2 - \frac{m\,\dot{a}(t)\,\dot{q}(t)}{\hbar\, a(t)} \times [x - q(t)]. \qquad (2.1.3.12)$$

Considering the eqs. (2.1.2.1a,b) and (2.1.3.1a), let us write V_{qu} given by eq. (2.1.2.1e,f) in terms of potencies of [x - q(t)]. Before, we calculate the following derivations:

$$\frac{\partial \varphi}{\partial x} = \frac{\partial}{\partial x}(\ [2\,\pi\, a^2(t)]^{-1/4} \times exp\ \{-\ \frac{[x - q(t)]^2}{4\,a^2(t)}\}\ ) =$$

$$= [2\,\pi\, a^2(t)]^{-1/4} \times exp\{-\frac{[x-q(t)]^2}{4\,a^2(t)}\} \times \frac{\partial}{\partial x}\{-\frac{[x-q(t)]^2}{4\,a^2(t)}\} \quad \rightarrow$$

$$\frac{\partial \varphi}{\partial x} = -[2\,\pi\, a^2(t)]^{-1/4} \times exp\,\{-\frac{[x-q(t)]^2}{4\,a^2(t)}\} \times \{\frac{[x-q(t)]}{2\,a^2(t)}\}\ ,$$

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial}{\partial x}(-[2\,\pi\, a^2(t)]^{-1/4} \times exp\,\{-\frac{[x-q(t)]^2}{4\,a^2(t)}\} \times \{\frac{[x-q(t)]}{2\,a^2(t)}\} =$$

$$= -[2\,\pi\, a^2(t)]^{-1/4} \times exp\,\{-\frac{[x-q(t)]^2}{4\,a^2(t)}\} \times \frac{\partial}{\partial x}\{\frac{[x-q(t)]}{2\,a^2(t)}\} -$$

$$-[2\,\pi\, a^2(t)]^{-1/4} \times exp\,\{-\frac{[x-q(t)]^2}{4\,a^2(t)}\} \times \frac{\partial}{\partial x}\{\frac{-[x-q(t)]^2}{4\,a^2(t)}\} \quad \rightarrow$$

$$\frac{\partial^2 \varphi}{\partial x^2} = -[2\,\pi\, a^2(t)]^{-1/4} \times exp\,\{-\frac{[x-q(t)]^2}{4\,a^2(t)}\} \times \{\frac{[x-q(t)]}{2\,a^2(t)}\} \times \frac{1}{2\,a^2(t)} +$$

$$+ [2\,\pi\, a^2(t)]^{-1/4} \times exp\,\{-\frac{[x-q(t)]^2}{4\,a^2(t)}\}\} \times \{\frac{[x-q(t)]^2}{4\,a^2(t)}\} \quad \rightarrow$$

$$\frac{1}{\varphi}\frac{\partial^2 \varphi}{\partial x^2} = \frac{[x-q(t)]^2}{4\,a^4(t)} - \frac{1}{2\,a^2(t)}\,. \qquad (2.1.3.13)$$

Substituting the relation (2.1.3.13) in the equation (2.1.2.1e), taking into account the expression (2.1.3.8), and considering the identity of polynomial, results:

$$V_{qu}(x,\,t) = -\frac{\hbar^2}{2\,m}\{\frac{[x-q(t)]^2}{4\,a^2(t)} - \frac{1}{2\,a^2(t)}\} =$$

$$V_{qu}(x,\,t) = V_{qu}[q(t),\,t] + V'_{qu}[q(t),\,t] \times [x-q(t)] + \frac{1}{2}V''_{qu}[q(t),\,t] \times [x - q(t)]^2 \quad \rightarrow$$

$$V_{qu}[q(t),\,t] = \frac{\hbar^2}{4\,m\,a^2(t)},\ \ V'_{qu}[q(t),\,t] = 0,\ \ V''_{qu}[q(t),\,t] = -\frac{\hbar^2}{4\,m\,a^4(t)} \quad \rightarrow$$

$$V_{qu}[q(t),t] = \frac{\hbar^2}{4\,m\,a^2(t)} - \frac{\hbar^2}{8\,m\,a^4(t)} \times [x-q(t)]^2\,. \qquad (2.1.3.14)$$

Now, let us write V_{BBM} given by eq. (2.1.2.4b) in terms of the potencies of [x, - q(t)]. Initially using the eqs. (2.1.3.1a) and (2.1.3.9) and considering the identity of polynomial [remembering that $\ell n\,(ab) = \ell n\,a + \ell n\,b$, $\ell n\,exp(x) = x$ and $\ell n\,a^r = r\,\ell n\,a$], results:

$$V_{BBM}(x\ ,\ t) = \frac{\hbar\,\lambda}{2}\,\ell n\,\varrho = \frac{\hbar\,\lambda}{2} \times ([2\,\pi\,a^2(t)]^{-1/2} \times exp\ \{-\ \frac{[x - q(t)]^2}{2\,a^2(t)}\}\ ) =$$

$$= -\,(\hbar\,\lambda/4) \times \{\ell n\ [2\,\pi\,a^2(t) + \frac{[x - q(t)]^2}{2\,a^2(t)}\} =$$

$$= V_{BBM}[q(t),\,t] + V'_{BBM}[q(t),\,t] \times [x - q(t)]v + \frac{1}{2}\,V''_{BBM}[q(t),\,t] \times [x - q(t)]^2 \quad \rightarrow$$

$$V_{BBM}[q(t),\,t] = -\,(\hbar\,\lambda/4) \times \{\ell n\ [2\,\pi\,a^2(t)]\},\quad V'_{BB-M}[q(t),\,t] = 0\ ,$$

$$V''_{BBM}[q(t),\,t] = -\ \frac{\hbar\,\lambda}{4\,a^2(t)} \quad \rightarrow$$

$$V_{BBM}[q(t),\,t] = -\,(\hbar\,\lambda/4) \times \{\ell n\ [2\,\pi\,a^2(t)]\} - \frac{\hbar\,\lambda}{4\,a^2(t)} \times [x - q(t)]^2\ . \qquad (2.1.3.15)$$

Using the eqs. (2.1.1.5), (2.1.2.1c-e) and (2.1.2.4b), results:

$$-\,\hbar\ \frac{\partial S}{\partial t} = [-\ \frac{\hbar^2}{2\,m}\,\frac{1}{\varphi}\,\frac{\partial^2\,\varphi}{\partial x^2} + \frac{m}{2}\,(\frac{\hbar}{m}\,\frac{\partial S}{\partial x})^2] + V(x\ ,\ t) - \frac{\hbar\,\lambda}{2}\,\ell n\ (\varphi^2) \quad \rightarrow$$

$$\hbar\ \frac{\partial S}{\partial t} + \frac{m}{2}\,v_{qu}^2 + V(x,\,t) + V_{qu}(x,\,t) - V_{BBM}(x,\,t) = 0\ . \qquad (2.1.3.16)$$

Inserting the eqs. (2.1.3.5,9,14,15) into eq. (2.1.3.16), we obtain:

$$\hbar\ \{\dot{S}_0(t) + \frac{m\,\ddot{q}(t)}{\hbar} \times [x - q(t)] - \frac{m\,\dot{q}^2(t)}{\hbar} + \frac{m}{2\,\hbar}\,[\ \frac{\ddot{a}}{a(t)} - \frac{\dot{a}^2(t)}{a^2(t)}\ ] \times [x - q(t)]^2 -$$

$$-\ \frac{m\,\dot{a}(t)\,\dot{q}(t)}{\hbar\,a(t)} \times [x - q(t)]\ \}\ + \frac{1}{2}\,\{\ \frac{\dot{a}(t)}{a(t)}\,[x - q(t)] + \dot{q}(t)\ \}^2 +$$

$$+\ V[q(t),\,t] + V'[q(t),\,t] \times \frac{1}{2}\,V''[q(t),\,t] \times [x - q(t)]^2 +$$

$$+\ \frac{\hbar^2}{4\,m\,a^2(t)} - \frac{\hbar^2}{8\,m\,a^4(t)} \times [x - q(t)]^2 + \frac{\hbar\,\lambda}{2} \times \ell n\ [\pi\,a^2(t)] + \frac{\hbar\,\lambda}{2\,a^2(t)} \times [x - q(t)]^2 = 0\ .$$

Since $(x - q)^o = 1$, we can gather together the above expression in potencies of $(x - q)$, obtaining:

$$\{\hbar\,\dot{S}_o(t) - \frac{1}{2}\,m\,\dot{q}^2(t) + V[q(t),\,t] + \frac{\hbar^2}{4\,m\,a^2} + \frac{\hbar\,\lambda}{4}\,\ell n\ [\pi\,a^2(t)]\} \times [x - q(t)]^0 +$$

$$+\ \{m\,\ddot{q}(t) + V'[q(t),\,t]\} \times [x - q(t)] +$$

$$\{\ \frac{m}{2}\,\frac{\ddot{a}(t)}{a(t)} + \frac{1}{2}\,V''[q(t),\,t] - \frac{\hbar^2}{8\,m\,a^4(t)}\,\frac{\hbar\,\lambda}{2\,a^2(t)}\ \} \times [x - (q)t)]^2 = 0\ . \qquad (2.1.3.17)$$

As the above relation is an identically null polynomium, the coefficients of the potencies must be all equal to zero, that is:

$$\dot{S}_o(t) = \frac{1}{\hbar} \{ \frac{1}{2} m \dot{q}^2 - V[q(t), t] - \frac{\hbar^2}{4\, m\, a^2(t)} - \frac{\hbar\, \lambda}{4}\, \ell n\, [\pi\, a^2(t)]\} \,, \quad (2.1.3.18)$$

$$\ddot{q} + \frac{1}{m} V'[q(t), t] = 0 \,, \quad (2.1.3.19)$$

$$\ddot{a}(t) + \frac{1}{m} V''[q(t), t] - \frac{\hbar^2}{4\, m^2\, a^3(t)} + \frac{\hbar\, \lambda}{m\, a(t)} = 0 \,. \quad (2.1.3.20)$$

Assuming that the following initial conditions are obeyed:

$$q(0) = x_o \,, \quad \dot{q}(0) = v_o \,, \quad a(0) = a_o \,, \quad \dot{a}(0) = b_o \,, \quad (2.1.3.21a\text{-}d)$$

and that [see eqs.(2.1.2.1c,d) and (2.1.3.11b)]:

$$S_o(0) = \frac{m\, v_o\, x_o}{\hbar} \,, \quad (2.1.3.22)$$

the integration of the expression (2.1.3.18) will be given by:

$$S_o(t) = \frac{1}{\hbar} \int_o^t dt' \, \{ \frac{1}{2} m \dot{q}^2(t') - V[q(t'), t'] - \frac{\hbar^2}{4\, m\, a^2(t')} - \frac{\hbar\, \lambda}{4}\, \ell n\, [\pi\, a^2(t')]\} +$$

$$- \frac{m\, v_o\, x_o}{\hbar} \,. \quad (2.1.3.23)$$

Taking into account the expressions (2.1.3.11a,b) in the equation (2.1.3.23) results:

$$S(x, t) = \frac{1}{\hbar} \int_o^t dt' \, \{ \frac{1}{2} m \dot{q}^2(t') - V[q(t'), t'] - \frac{\hbar^2}{4\, m\, a^2(t')} - \frac{\hbar\, \lambda}{4}\, \ell n\, [\pi\, a^2(t')]\} +$$

$$+ \frac{m\, v_o\, x_o}{\hbar} + \frac{m\, \dot{q}(t)}{\hbar} \times [x - q(t)] + \frac{m\, \dot{a}(t)}{2\, \hbar\, a(t)} \times [x - q(t)]^2 \,. \quad (2.1.3.24)$$

This result obtained above permit us, finally, to obtain the wave packet for the *BB−ME*. Indeed, considering the equations (2.1.3.1b) and (2.1.3.23), we get: [4]

$$\Psi(x, t) = [2\, \pi\, a^2(t)]^{-1/4}\, exp\, [\, \frac{i\, m}{2\, \hbar} \frac{\dot{a}(t)}{a(t)} - \frac{1}{4\, a^2(t)}\,] \times [x - q(t)]^2 \} \times$$

$$\times\, exp\, \{ \frac{i\, m\, \dot{q}(t)}{\hbar} \times [x - q(t)] + \frac{i\, m\, v_o\, x_o}{\hbar} \} \times$$

$$\times\, exp\, [\, \frac{i}{\hbar} \int_o^t dt' \, \{ \frac{1}{2} m \dot{q}^2(t') - V[q(t'), t'] -$$

$$- \frac{\hbar^2}{4\, m\, a^2(t')} - \frac{\hbar\, \lambda}{4}\, \ell n\, [\pi\, a^2(t')]\}\,] \,. \quad (2.1.3.25)$$

2.1.4. Calculation of the Feynman Propagator of the Linearized Bialynicki-Birula-Mycielski Equation along a Classical Trajetory

The looked for Feynman-de Broglie-Bohm propagator of the linearized *BBM−E* along a classical trajectory, will be calculated using the eqs. (1.3) and (2.1.3.25). However, in the eq. (1.3), we will put with no loss of generality, $t_o = 0$. Thus: [4]

$$\Psi(x, t) = \int_{-\infty}^{+\infty} K(x, x_o, t, 0)\, \Psi(x_o, 0)\, dx_o \,. \quad (2.1.4.1)$$

Initially let us define the normalized quantity:

$$\Phi(v_o, x, t) = (2\, \pi\, a_o^2)^{1/4}\, \Psi(v_o, x, t) \,, \quad (2.1.4.2)$$

which satisfies the following *completeness relation*: [3]

$$\int_{-\infty}^{+\infty} dv_o\, \Phi^*(v_o, x, t)\, \Phi(v_o, x', t) = \left(\frac{2\, \pi\, \hbar}{m}\right) \delta(x - x') \,. \quad (2.1.4.3)$$

Considering the eqs.(2.1.1.1), (2.1.2.1a,b) and (2.1.4.2,3), we get:

$$\Psi^*(x, t) \times \Psi(x, t) = \varphi^2(x, t) = \varrho(x, t) \,, \quad (2.1.4.4)$$

$$\Phi^*(v_o, x, t)\, \Psi(v_o, x, t) =$$

$$= (2\, \pi\, a_o^2)^{1/4}\, \Psi^*(v_o, x, t)\, \Psi(v_o, x, t) = (2\, \pi\, a_o^2)^{1/4}\, \varrho(v_o, x, t) \ \to$$

$$\varrho(v_o, x, t) = (2\, \pi\, a_o^2)^{-1/4}\, \Phi^*(v_o, x, t)\, \Psi(v_o, x, t) \,. \quad (2.1.4.5)$$

On the other side, substituting the eq. (2.1.4.5) into eq. (2.1.2.2), integrating the result and using the expressions (2.1.3.1a) and (2.1.4.2) results [remembering that $\frac{\partial}{\partial x}\frac{\partial}{\partial t} = \frac{\partial}{\partial t}\frac{\partial}{\partial x}$, $\Psi^*\, \Psi(\pm\infty) \to 0$, and the integration for parts]:

$$\frac{\partial(\Phi^*\, \Psi)}{\partial t} + \frac{\partial(\Phi^*\, \Psi\, v_{qu})}{\partial x} = 0 \ \to$$

$$\frac{\partial}{\partial t} \int_{-\infty}^{+\infty} dx\, \Phi^*\, \Psi + \int_{-\infty}^{+\infty} \frac{\partial\, (\Phi^*\, \Psi\, v_{qu})}{\partial x}\, dx =$$

$$= \frac{\partial}{\partial t} \int_{-\infty}^{+\infty} dx\, \Phi^*\, \Psi + (\, \Phi^*\, \Psi\, v_{qu})\,)\, |_{-infty}^{+infty} =$$

$$= \frac{\partial}{\partial t} \int_{-\infty}^{+\infty} dx\, \Phi^*\, \Psi + (2\, \pi\, a_o^2)^{1/4}\, (\, \Phi^*\, \Psi\, v_{qu})\,)\, |_{-infty}^{+infty} = 0 \ \to$$

$$\frac{\partial}{\partial t} \int_{-\infty}^{+\infty} dx\, \Phi^*\, \Psi + (\Phi^*\, \Psi\, v_{qu})|_{-\infty}^{+\infty} =$$

$$= \frac{\partial}{\partial t} \int_{-\infty}^{+\infty} dx\, \Phi^*\, \Psi + (2\, \pi\, a_o^2)^{1/4}\, (\Psi^*\, \Psi\, v_{qu})|_{-\infty}^{+\infty} = 0 \ \to$$

$$\frac{\partial}{\partial t} \int_{-\infty}^{+\infty} dx\, \Phi^* \Psi = 0\,. \qquad (2.1.4.6)$$

The eq. (2.1.4.6) shows that the integration is time independent. Consequently:

$$\int_{-\infty}^{+\infty} dx'\, \Phi^*(v_o, x', t)\, \Psi(x', t) = \int_{-\infty}^{+\infty} dx_o\, \Phi^*(v_o, x_o, 0)\, \Psi(x_o, 0)\,. \qquad (2.1.4.7)$$

Multiplying the eq. (2.1.4.7) by $\Phi(v_o, x, t)$ and integrating over v_o and using the eq. (2.1.4.4), we will obtain [remembering that $\int_{-\infty}^{+\infty} dx'\, f(x')\, \delta(x' - x) = f(x)$]:

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dv_o\, dx'\, \Phi(v_o, x, t)\, \Phi^*(v_o, x', t)\, \Psi(x', t) =$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dv_o\, dx_o\, \Phi(v_o, x, t)\, \Phi^*(v_o, x_o, 0)\, \Psi(x_o, 0) \;\rightarrow$$

$$\int_{-\infty}^{+\infty} dx' \left(\frac{2\pi\hbar}{m}\right) \delta(x' - x)\, \Psi(x', t) = \left(\frac{2\pi\hbar}{m}\right) \Psi(x, t) =$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dv_o\, \Phi(v_o, x, t)\, \Phi^*(v_o, x_o, 0)\, \Psi(x_o, 0) \;\rightarrow$$

$$\Psi(x, t) = \int_{-\infty}^{+\infty} \left\{ \left(\frac{m}{2\pi\hbar}\right) \int_{-\infty}^{+\infty} dv_o\, \Phi(v_o, x, t) \times\right.$$

$$\left. \times\, \Phi^*(v_o, x_o, 0) \right\} \Psi(x_o, 0)\, dx_o\,. \qquad (2.1.4.8)$$

Comparing the eqs. (2.1.4.1,8), we have:

$$K(x, x_o, t) = \left(\frac{m}{2\pi\hbar}\right) \int_{-\infty}^{+\infty} dv_o\, \Phi(v_o, x, t)\, \Phi^*(v_o, x_o, 0)\,. \qquad (2.1.4.9)$$

Substituting the eqs. (2.1.3.25) and (2.1.4.2) in the equation (2.1.4.9), we obtain the Feynman Propagator of the linearized Bialynicki-Birula-Mycielski Equation along a classical trajetory, that we were looking for, that is [remembering that $\Phi^*(v_o, x_o, 0) = exp\left(-\frac{i\, m\, v_o\, x_o}{\hbar}\right)$]:

$$K(x, x_o; t) = \frac{m}{2\pi\hbar} \int_{-\infty}^{+\infty} dv_o \sqrt{\frac{a_o}{a(t)}} \times$$

$$\times\, exp \left\{\left[\frac{i}{2} \frac{m}{\hbar} \frac{\dot{a}(t)}{a(t)} - \frac{1}{4\, a^2(t)} \right] \times [x - q(t)]^2 \right\} \times$$

$$\times\, exp \left\{ \frac{i\, m\, \dot{q}(t)}{\hbar} \times [x - q(t)] \right\} \times$$

$$\times\, exp \left(\frac{i}{\hbar} \int_o^t dt' \left\{ \frac{1}{2} m\, \dot{q}^2(t') - V[q(t'), t'] - \frac{\hbar^2}{4\, m\, a^2(t')} - \frac{\hbar\lambda}{4} \ell n\, [\pi\, a^2(t')] \right\} \right), \quad (2.1.4.10)$$

where $q(t)$ and $a(t)$ are solutions of the differential equations given by the eqs.(2.1.3.19,20).

Finally, it is important to note that putting $\lambda = 0$ and $V[q(t'), t'] = 0$ into eq. (2.1.4.10) and eqs. (2.1.3.19,20) we obtain the free Feynman propagator. [2,4]

2.2. The Bateman-Caldirola-Kanai Equation

In 1931/1941/1948, H. Bateman, [10], P. Caldirola [11] and E. Kanai [12] proposed a non-linear Schrödinger equation, to represent time dependent physical systems, defined by:

$$i\,\hbar\,\frac{\partial \Psi(x,\,t)}{\partial t} = -\,\frac{\hbar^2}{2\,m}\,exp\,(-\,\lambda\,t)\times\frac{\partial^2\,\Psi(x,\,t)}{\partial x^2} +$$

$$+\,exp\,(\lambda\,t)\times V(x,\,t)\times\Psi(x,\,t)\,, \qquad (2.2.1)$$

where $\Psi(x,\,t)$ and $V(x,\,t)$ are, respectively, the wave function and the time dependent potential of the physical system in study, and λ is a constant.

2.2.1. The Wave Function of the Bateman-Caldirola-Kanai Equation

Putting the eqs. (2.1.1.1) and (2.1.1.2a,b) into the eq. (2.2.1), we have:

$$i\,\hbar\,(\,\frac{\partial\varphi}{\partial t} + i\,\varphi\,\frac{\partial S}{\partial t}\,) = -\,\frac{\hbar^2}{2\,m}\,exp\,(-\,\lambda\,t)\times$$

$$\times\,[\,\frac{\partial^2\varphi}{\partial x^2} + 2\,i\,\frac{\partial S}{\partial x}\,\frac{\partial\varphi}{\partial x} + i\,\varphi\,\frac{\partial^2 S}{\partial x^2} - \varphi\,(\,\frac{\partial S}{\partial x})^2\,] + exp\,(\,\lambda\,t)\times V(x,\,t)\times\varphi\,, \qquad (2.2.1.2)$$

Separating the real and imaginary parts of the relation (2.2.1.2), results:

a) <u>imaginary part</u>

$$\frac{\hbar}{\varphi}\,\frac{\partial\varphi}{\partial t} = -\,\frac{\hbar^2}{2\,m}\,exp\,(-\,\lambda\,t)\times(\,2\,\frac{1}{\varphi}\,\frac{\partial S}{\partial x}\,\frac{\partial\varphi}{\partial x} + \frac{\partial^2 S}{\partial x^2}\,)\,, \qquad (2.2.1.3)$$

b) <u>real part</u>

$$-\,\hbar\,\frac{\partial S}{\partial t} = -\,\frac{\hbar^2}{2\,m}\,exp\,(-\,\lambda\,t)\times[\,\frac{1}{\varphi}\,\frac{\partial^2\varphi}{\partial x^2} - (\,\frac{\partial S}{\partial x})^2\,] + exp\,(\lambda\,t)\times V(x,\,t)\,. \qquad (2.2.1.4)$$

2.2.2. Dynamics of the Bateman-Caldirola-Kanai Equation

Now, let us see the correlation between the expressions (2.2.1.3-4) and the traditional equations of the Ideal Fluid Dynamics [8] a) *Continuity Equation*, b) *Euler's equation* (for conservative systems) or b' *Navier-Stokes equation* (for non-conservative systems). Thus, putting the eqs. (2.1.2.1a-f) into the eqs. (2.2.1.3-4), and using the same operational protocol of the item (2.1.2), we obtain: [4]

$$\frac{\partial\varrho}{\partial t} + exp\,(-\,\lambda\,t)\times(\,\varrho\,\frac{\partial v_{qu}}{\partial x} + v_{qu}\,\frac{\partial\varrho}{\partial x}\,) = 0 \;\;\rightarrow$$

$$\frac{\partial\varrho}{\partial x} + \frac{\partial(\varrho\,v_{BCK})}{\partial x} = 0, \qquad (2.2.2.1a)$$

where:

$$v_{BCK} = exp\,(-\,\lambda\,t)\times v_{qu}, \qquad (2.2.2.1b)$$

is the *Bateman-Caldirola-Kanai quantum velocity*.

We observe that the eq. (2.2.2.1a) represents the *Continuity Equation* or *Mass Conservation Law* of the Fluid Dynamics. We must note that this expression also indicates <u>coerence</u> of the considered physical system represented by the Bateman-Caldirola-Kanai Equation (*BCK−E*) [eq. (2.2.1)].

Now, let us obtained another dynamic equation of the *B−C−KE*. So, differentiating the eq. relation (2.2.1.4) with respect *x* and using the eqs. (2.1.2.1a-e), we obtain:

$$- \hbar \frac{\partial^2 S}{\partial x\, \partial t} = - \frac{\hbar^2}{2m} exp\,(-\lambda\, t) \times \frac{\partial}{\partial x} [\frac{1}{\varphi} \frac{\partial^2 \varphi}{\partial x^2} - (\frac{\partial S}{\partial x})^2] + exp\,(\lambda\, t) \times \frac{\partial V(x, t)}{\partial x} \rightarrow$$

$$\frac{\partial}{\partial t} (\frac{\hbar}{m} \frac{\partial S}{\partial x}) =$$

$$= - exp\,(-\lambda\, t) \times \frac{\partial}{\partial x} (\frac{\hbar^2}{2\, m^2} \frac{1}{\varphi} \frac{\partial^2 \varphi}{\partial x^2}) - \frac{exp\,(-\lambda\, t)}{2} \frac{\partial}{\partial x} (\frac{\hbar}{m} \frac{\partial S}{\partial x})^2 - \frac{exp\,(\lambda\, t)}{m} \frac{\partial V(x, t)}{\partial x} \rightarrow$$

$$\frac{\partial v_{qu}}{\partial t} = - \frac{exp\,(-\lambda\, t)}{m} \times \frac{\partial V_{qu}}{\partial x} - exp(-\lambda\, t) \times v_{qu} \times \frac{\partial v_{qu}}{\partial x} - \frac{exp\,(\lambda\, t)}{m} \times \frac{\partial V(x, t)}{\partial x}\,. \qquad (2.2.2.2)$$

Multiplying the eq. (2.2.2.2) by $exp\,(-\lambda\, t)$, using the eq. (2.2.2.1b) and its temporal derivate, we obtain:

$$exp\,(-\lambda\, t) \frac{\partial v_{qu}}{\partial t} =$$

$$= - \frac{exp\,(-2\,\lambda\, t)}{m} \times \frac{\partial V_{qu}}{\partial x} - exp(-\lambda\, t) \times v_{qu} \times \frac{\partial [exp\,(-\lambda\, t)\, v_{qu}]}{\partial x} - \frac{1}{m} \times \frac{\partial V(x, t)}{\partial x} \rightarrow$$

$$\frac{\partial v_{BCK}}{\partial t} + v_{BCK} \frac{\partial v_{BCK}}{\partial x} + \frac{1}{m} \times \frac{\partial}{\partial x} (V + V_{BCK}) = - \lambda\, v_{BCK}\,, \qquad (2.2.2.3a)$$

where:

$$V_{BCK}(x, t) = exp\,(-2\,\lambda\, t)\, V_{qu}(x, t)\,, \qquad (2.2.2.3b)$$

is the *Bateman-Caldirola-Kanai quantum potential*.

We observe that, although the eq. (2.2.2.3a) had the aspect of the *Navier-Stokes equation* [8], the same represent a conservative system, since when $\lambda \rightarrow \infty$, then v_{BCK} and $V_{BCK} \rightarrow 0$, by eqs. (2.2.2.1b; 2.2.2.3b).

Considering the *substantive differentiation* (local plus convective) or *hydrodynamic differentiation*, given by the eqs. (2.1.2.5a,b) and inserting into eq. (2.2.2.3a), results:

$$m \frac{d^2 x}{dt^2} + m\, \lambda\, v_{BCK}(x, t) = - \frac{\partial}{\partial x} [\, V(x, t) + V_{BCK}(x, t)\,], \qquad (2.2.2.4)$$

has a form of the *Dissipative Second Newton Law*, being the second term of the second member the *Bateman-Caldirola-Kanai force*.

2.2.3 The Quantum Wave Packet of the Linearized Bateman-Caldirola-Kanai Equation along a Classical Trajetory

In order to find the quantum wave packet of the linearized Bateman-Caldirola-Kanai Equation (*BCK−E*) along a classical trajetory, we calculate the v_{BCK} given by the eq. (2.2.2.1b). So, using the same operational protocol of the item (2.1.3), we have: [4]

$$v_{BCK}(x, t) \equiv \frac{dx(t)}{dt} = \frac{\dot{a}(t)}{a(t)} [x - q(t)] + \dot{q}(t). \qquad (2.2.3.1)$$

We observe that the integration of the eq. (2.2.3.1) given the *bohmnian quantum trajectory* of the physical system considered represented by the eq. (2.2.1).

To obtain the quantum wave packet [$\Psi(x\ ,\ t)$] of the *BCK−E* given by eq. (2.2.1), let us expand the functions S(x, t), V(x t), V_{qu}(x, t) and V_{BCK}(x, t) around of $q(t) = <x>$ up to *second Taylor order*. [9] In this way, we have: [4]

$$S(x, t) = S[q(t), t] + S'[q(t), t] \times [x - q(t)] + \frac{S''[q(t)\ t]}{2} \times [x - q(t)]^2, \qquad (2.2.3.2)$$

$$V(x, t) = V[q(t), t] + V'[q(t), t] \times [x - q(t)] + \frac{V''[q(t), t]}{2} \times [x - q(t)]^2, \qquad (2.2.3.3)$$

$$V_{qu}(x, t) = V_{qu}[q(t), t] + V'_{qu}[q(t), t] \times [x - q(t)] +$$

$$+ \frac{1}{2} V''_{qu}[q(t), t] \times [x - q(t)]^2, \qquad (2.2.3.4)$$

$$V_{BCK}(x, t) = V_{BCK}[q(t), t] + V'_{BCK}[q(t), t] \times [x - q(t)] +$$

$$+ \frac{1}{2} V''_{BCK}[q(t), t] \times [x - q(t)]^2, \qquad (2.2.3.5)$$

$$S(x, t) = S_0(t) + \frac{m\, \dot{q}(t)}{\hbar} [x - q(t)] + \frac{m\, \dot{a}(t)}{2\, \hbar\, a(t)} [x - q(t)]^2 , \qquad (2.2.3.6)$$

$$\frac{\partial S}{\partial t} = \dot{S}_0(t) + \frac{\partial}{\partial t} \{ \frac{m\, \dot{q}(t)}{\hbar} [x - q(t)] \} + \frac{\partial}{\partial t} \{ \frac{m\, \dot{a}(t)}{2\, \hbar\, a(t)} [x - q(t)]^2 \} \quad \rightarrow$$

$$\frac{\partial S}{\partial t} = \dot{S}_0(t) + \frac{\ddot{q}(t)}{\hbar} [x - q(t)] - \frac{m\, \dot{q}^2(t)}{\hbar} +$$

$$+ \frac{m}{2\, \hbar} [\ \frac{\ddot{a}(t)}{a(t)} - \frac{\dot{a}^2(t)}{a^2(t)} \times [x - qt(t)]^2 - \frac{m\, \dot{a}(t)\, \dot{q}(t)}{\hbar\, a(t)} \times [x - q(t)] . \qquad (2.2.3.7)$$

where:

$$S_0(t) \equiv S[q(t), t] , \qquad (2.2.3.8)$$

are the *quantum action* [see eq. (2.1.3.11b)].

Now, let us write $V_{qu}(x, t)$ and $V_{BCK}(x, t)$ in terms of potencies of [x - q(t)]. So, using the eqs. (2.1.2.1e), (2.1.3.1b), (2.1.3.13), (2.2.2.3b), (2.2.3.4) and (2.2.3.5), results:

$$V_{qu}(x, t) = \frac{\hbar^2}{4\, m\, a^2(t)} - \frac{\hbar^2}{8\, m\, a^4(t)} \times [x - q(t)]^2 , \qquad (2.2.3.9)$$

$$V_{BCK}(x, t) = exp\, (- 2\, \lambda\, t) \times \{ \frac{\hbar^2}{4\, m\, a^2(t)} - \frac{\hbar^2}{8\, m\, a^4(t)} \times [x - q(t)]^2 \} . \qquad (2.2.3.10)$$

Using the eqs. (2.1.2.1c,d), (2.1.2.1e), (2.2.1.4), (2.2.2.1b) and (2.2.2.3b), we obtain:

$$- \hbar \frac{\partial S}{\partial t} = - \frac{\hbar^2}{2\, m}\, exp\,(- \lambda\, t) \times [\, \frac{1}{\varphi} \frac{\partial^2 \varphi}{\partial x^2} + (\, \frac{\partial S}{\partial x})^2] + exp\,(\lambda\, t) \times V(x\,,\, t) \quad \rightarrow$$

$$\hbar \frac{\partial S}{\partial t} + exp\,(- \lambda\, t) \times [\, \frac{m}{2} v_{qu}^2 + V_{qu}(x,\, t)\,] + exp\,(\lambda\, t) \times V(x,\, t) = 0 \quad \rightarrow$$

$$\hbar \frac{\partial S}{\partial t} + exp\,(\lambda\, t) \times [\, \frac{m}{2} [exp\,(- \lambda\, t)\, v_{qu}]^2 + exp\,(- 2\, \lambda\, t) \times V_{qu}(x,\, t) + V(x,\, t)\,] = 0 \quad \rightarrow$$

$$\hbar \frac{\partial S}{\partial t} + exp\,(\lambda\, t) \times [\, \frac{m}{2} v_{BCK}^2 + V(x,\, t) + V_{BCK}(x,\, t)\,] = 0\,. \qquad (2.2.3.11)$$

Inserting the eqs. (2.2.3.1), (2.2.3.3), (2.2.3.7) and (2.2.3.10) into eq. (2.2.3.11), we obtain:

$$\hbar\, \{\dot{S}_0(t) + \frac{m\, \ddot{q}(t)}{\hbar} \times [x - q(t)] - \frac{m\, \dot{q}^2(t)}{\hbar} + \frac{m}{2\, \hbar} [\, \frac{\ddot{a}}{a(t)} - \frac{\dot{a}^2(t)}{a^2(t)}\,] \times [x - q(t)]^2 -$$

$$- \frac{m\, \dot{a}(t)\, \dot{q}(t)}{\hbar\, a(t)} \times [x - q(t)]\, \} +$$

$$+ exp\,(\lambda\, t) \times (\, \frac{m}{2} \{\, \frac{\dot{a}(t)}{a(t)} [x - q(t)] + \dot{q}(t)\, \}^2\,) +$$

$$+ exp\,(\lambda\, t) \times \{\, V[q(t),\, t] + V'[q(t),\, t] \times \frac{1}{2} V''[\mathrm{q(t), t}] \times [\mathrm{x - q(t)}]^2\, \} +$$

$$+ exp\,(- \lambda\, t) \times \{\, \frac{\hbar^2}{4\, m\, a^2(t)} - \frac{\hbar^2}{8\, m\, a^4(t)} \times [x - q(t)]^2\, \}\,. \qquad (2.2.3.12)$$

Since $(x - q)^o = 1$, we can gather together the above expression in potencies of $(x - q)$, obtaining:

$$(\, \hbar\, \dot{S}_o(t) - m\, \dot{q}^2(t) + exp\,(\lambda\, t) \times \{\, \frac{m}{2} \dot{q}^2(t) + V[q(t),\, t]\, \} + \frac{exp\,(- \lambda\, t)\, \hbar^2}{4\, m\, a^2}\,) \times [x - q(t)]^0 +$$

$$+ \{m\, \ddot{q}(t) + \frac{m\, \dot{a}(t)\, \dot{q}(t)}{a(t)} [exp\,(\lambda\, t) - 1] + exp\,(\lambda\, t)\, V'[q(t),\, t]\} \times [x - q(t)] +$$

$$+ \{\, \frac{m}{2} [\, \frac{\ddot{a}(t)}{a(t)} - \frac{\dot{a}^2(t)}{a^2(t)}\,] + \frac{exp\,(\lambda\, t)}{2} (\, \frac{m\, \dot{a}^2(t)}{a^2(t)} +$$

$$+ V''[\mathrm{q(t), t}]\,) - \frac{\mathrm{exp\,(-\, \lambda\, t)}\, \hbar^2}{\mathrm{8\, m\, a^4(t)}} \frac{\hbar\, \lambda}{\mathrm{2\, a^2(t)}}\, \} \times [\mathrm{x - (q)t}]^2 = 0\,. \qquad (2.2.3.13)$$

As the above relation is an identically null polynomium, the coefficients of the potencies must be all equal to zero, that is:

$$\dot{S}_o(t) = \frac{1}{\hbar} (\, m\, \dot{q}^2 - exp\,(\lambda\, t) \times \{\, \frac{m}{2} \dot{q}(t) + V[q(t),\, t]\, \} - \frac{exp\,(- \lambda\, t)\, \hbar^2}{4\, m\, a^2(t)}\,)\,, \qquad (2.2.3.14)$$

$$\ddot{q} + \frac{\dot{a}(t)\,\dot{q}(t)}{a(t)}\,[exp\,(\lambda\,t) - 1] + exp\,(\lambda\,t) \times \frac{V'[q(t),\,t]}{m} = 0\,, \quad (2.2.3.15)$$

$$\ddot{a}(t) - \frac{\dot{a}^2(t)}{a(t)} + exp\,(\lambda\,t) \times \{\, \frac{\dot{a}^2(t)}{a^2(t)} + \frac{V''[q(t),\,t]}{m} \,\} \times a(t) = \frac{exp\,(-\,\lambda\,t)\,\hbar^2}{4\,m^2\,a^3(t)}\,. \quad (2.2.3.16)$$

Assuming that the following initial conditions are obeyed:

$$q(0) = x_o\,, \quad \dot{q}(0) = v_o\,, \quad a(0) = a_o\,, \quad \dot{a}(0) = b_o\,, \quad (2.2.3.17a\text{-}d)$$

and that [see eqs.(2.1.2.1c,d) and (2.1.4.11b)]:

$$S_o(0) = \frac{m\,v_o\,x_o}{\hbar}\,, \quad (2.2.3.18)$$

the integration of the expression (2.2.3.14) will be given by:

$$S_o(t) = \frac{1}{\hbar}\int_o^t dt'\,\{\, m\,\dot{q}^2(t') - exp\,(\lambda\,t') \times (\, \frac{m}{2}\dot{q}(t') + V[q(t'),\,t']\,) -$$

$$-\,\frac{exp\,(-\,\lambda\,t')\,\hbar^2}{4\,m\,a^2(t')}\,\} + \frac{m\,v_o\,x_o}{\hbar}\,. \quad (2.2.3.19)$$

Taking into account the expressions (2.2.3.19) in the equation (2.2.3.6) results:

$$S(x,\,t) = \frac{1}{\hbar}\int_o^t dt'\,(\, m\,\dot{q}^2(t') - exp\,(\lambda\,t') \times \{\, \frac{m}{2}\dot{q}(t') +$$

Erro!

$$+\,\frac{m\,\dot{q}(t)}{\hbar} \times [x - q(t)] + \frac{m\,\dot{a}(t)}{2\,\hbar\,a(t)} \times [x - q(t)]^2\,. \quad (2.2.3.20)$$

This result obtained above permit us, finally, to obtain the wave packet for the *BCK−E*. Indeed, considering the eqs. (2.1.1.1), (2.1.3.1b) and (2.2.3.20), we get: [4]

$$\Psi(x,\,t) = [2\,\pi\,a^2(t)]^{-\,1/4} \times exp\,\{\,[\,\frac{i\,m}{2\,\hbar}\,\frac{\dot{a}(t)}{a(t)} - \frac{1}{4\,a^2(t)}\,] \times [x - q(t)]^2\} \times$$

$$\times\,exp\,\{\,\frac{i\,m\,\dot{q}(t)}{\hbar} \times [x - q(t)] + \frac{i\,m\,v_o\,x_o}{\hbar}\,\} \times$$

$$\times\,exp\,[\,\frac{i}{\hbar}\int_o^t dt'\,(\, m\,\dot{q}^2(t') - exp\,(\lambda\,t') \times \{\, \frac{m}{2}\dot{q}(t') +$$

$$+\,V[q(t'),\,t']\,\} - \frac{exp\,(-\,\lambda\,t')\,\hbar^2}{4\,m\,a^2(t')}. \quad (2.2.3.21)$$

2.2.4. Calculation of the Feynman Propagator of the Linearized Bateman-Caldirola-Kanai Equation along a Classical Trajetory

For to calculated the Feynman-de Broglie-Bohm propagator of the linearized *BCK−E* along a classical trajectory, will be following the same operational protocol of the item (2.1.4). Thus, by using the eqs. (2.1.4.1-3,5,9), we can write that: [4]

$$\Psi(x, t) = \int_{-\infty}^{+\infty} K(x, x_o;t, 0)\, \Psi(x_o, 0)\, dx_o\,, \quad (2.2.4.1)$$

$$\Phi(v_o, x, t) = (2\,\pi\, a_o^2)^{1/4}\, \Psi(v_o, x, t)\,, \quad (2.2.4.2)$$

$$\int_{-\infty}^{+\infty} dv_o\, \Phi^*(v_o, x', t)\, \Phi(v_o, x', t) = \left(\frac{2\, pi\, \hbar}{m} \right) \delta(x - x')\,, \quad (2.2.4.3)$$

$$\Psi(x, t) = \int_{-\infty}^{+\infty} \left\{ \left(\frac{m}{2\,\pi\,\hbar} \right) \int_{-\infty}^{+\infty} dv_o\, \Phi(v_o, x, t)\, \Phi^*(v_o, x_o, 0)\, \Psi(x_o, 0)\, \Psi(x_o, 0) \right\} \times$$

$$\times\, \Psi(x_o, 0)\, dx_o\,. \quad (2.2.4.4)$$

Comparing the eqs. (2.2.4.1,4), we have:

$$K(x, x_o, t) = \left(\frac{m}{2\,\pi\,\hbar} \right) \int_{-\infty}^{+\infty} dv_0\, \Phi(v_o, x, t)\, \Phi^*(v_o, x_o, 0)\,. \quad (2.2.4.5)$$

Substituting the eqs.(2.2.3.21) and (2.2.4.2) in the equation (2.2.4.5), we obtain the Feynman Propagator of the linearized Bateman-Caldirola-Kanai Equation along a classical trajetory, that we were looking for, that is [remembering that $\Phi^*(v_o, x_o, 0) = exp\,(- \frac{i\, m\, v_o\, x_o}{\hbar})$]:

$$K(x, x_o; t) = \left(\frac{m}{2\,\pi\,\hbar} \right) \int_{-\infty}^{+\infty} dv_o \sqrt{\frac{a_0}{a(t)}} \times$$

$$\times\, exp\, \left\{ \left[\frac{i}{2}\, \frac{m}{\hbar}\, \frac{\dot{a}(t)}{a(t)} - \frac{1}{4\, a^2(t)} \right] \times [x - q(t)]^2 \right\} \times$$

$$\times\, exp\, \left\{ \frac{i\, m\, \dot{q}(t)}{\hbar} \times [x - q(t)] \right\} \times$$

$$\times\, exp\, \left[\frac{i}{\hbar} \int_o^t dt' \left\{ m\, \dot{q}^2(t') - exp(\lambda\, t') \times \left(\frac{m}{2} \dot{q}(t') + V[q(t'), t'] \right) - \right. \right.$$

$$\left. \left. - \frac{exp\,(-\lambda\, t')\, \hbar^2}{4\, m\, a^2(t')} \right\} \right]\,, \quad (2.2.4.6)$$

where $q(t)$ and $a(t)$ are solutions of the differential equations given by the eqs.(2.2.3.15,16).

Finally, it is important to note that putting $\lambda = 0$ and $V[q(t'), t'] = 0$ into the eqs. (2.2.4.6) and (2.2.3.15,16) we obtain the free Feynman propagator. [2,4]

2.3. The Diósi-Halliwell-Nassar Equation

In 1998, L. Diósi and J. J. Halliwell [13] proposed a non-linear Schrödinger equation, to represent time dependent physical systems, defined by:

$$i\,\hbar\, \frac{\partial \Psi(x, t)}{\partial t} = - \frac{\hbar^2}{2\, m}\, \frac{\partial^2\, \Psi(x, t)}{\partial x^2} +$$

$$+ [\, V(x, t) + \lambda\, x\, X(t) - i\, \hbar\, (\frac{[x - q(t)]^2}{a^2} - \frac{\eta(t)\, [x - q(t)]}{a})\,] \times \Psi(x, t)\, , \quad (2.3.1)$$

where $\Psi(x, t)$ is a wave function which describes a given system, $X(t)$ is the position of classical particle submitted to a time dependent potential $V(x, t)$, $q(t) = \langle x(t) \rangle$, and λ and a are constants.

However, as the eq. (2.3.1) is not normalized, Nassar [14] considered $a(t)$ and $\eta(t) = a(t)/[x, q(t)]$, and proposed that:

$$i\, \hbar\, \frac{\partial \Psi(x, t)}{\partial t} = - \frac{\hbar^2}{2\, m} \frac{\partial^2\, \Psi(x, t)}{\partial x^2} +$$

$$+ [\, V(x, t) + \lambda\, x\, X(t) - \frac{i\, \hbar}{4\, \sigma} (\frac{[x - q(t)]^2}{a^2} - 1\,)\,] \times \Psi(x, t)\, , \quad (2.3.2)$$

where σ is a constant. The eq. (2.3.2) represents a *Schrödinger Equation for Continuous Quantum Measurements* or *Diósi-Halliwell-Nassar Equation* (*DHN−E*).

2.3.1. The Wave Function of the Diósi-Halliwell-Nassar Equation

Putting the eqs. (2.1.1.1) and (2.1.1.2a,b) into the eq. (2.3.2), we have: [4,15.3]

$$i\, \hbar\, (\frac{\partial \varphi}{\partial t} + i\, \varphi\, \frac{\partial S}{\partial t}) = - \frac{\hbar^2}{2\, m} [\, \frac{\partial^2 \varphi}{\partial x^2} + 2\, i\, \frac{\partial S}{\partial x} \frac{\partial \varphi}{\partial x} + i\, \varphi\, \frac{\partial^2 S}{\partial x^2} - \varphi\, (\frac{\partial S}{\partial x})^2\,] +$$

$$+ [\, V(x, t) + \lambda\, x\, X(t) - \frac{i\, \hbar}{4\, \sigma} (\frac{[x - q(t)]^2}{a^2(t)} - 1\,)\,] \times \varphi(x, t)\, , \quad (2.3.1.1)$$

Separating the real and imaginary parts of the eq. (2.3.1.1), results:

a) imaginary part

$$\frac{\hbar}{\varphi} \frac{\partial \varphi}{\partial t} = - \frac{\hbar^2}{2\, m} (2\, \frac{1}{\varphi} \frac{\partial S}{\partial x} \frac{\partial \varphi}{\partial x} + \frac{\partial^2 S}{\partial x^2}) - \frac{1}{4\, \sigma} (\frac{[x - q(t)]^2}{a^2(t)} - 1\,)\, , \quad (2.3.1.2)$$

b) real part

$$- \hbar\, \frac{\partial S}{\partial t} = - \frac{\hbar^2}{2\, m} [\, \frac{1}{\varphi} \frac{\partial^2 \varphi}{\partial x^2} - (\frac{\partial S}{\partial x})^2\,] + V(x, t) + \lambda\, x\, X(t))\, . \quad (2.3.1.3)$$

2.3.2. Dynamics of the Diósi-Halliwell-Nassar Equation

Now, let us see the correlation between the expressions (2.3.1.2-3) and the traditional equations of the Ideal Fluid Dynamics [8] a) *Continuity Equation*, b) *Euler's equation* (for conservative systems) or b') *Navier-Stokes equation* (for non-conservative systems). Thus, putting the eqs. (2.1.2.1a-f) into the eq. (2.3.1.2), and using the same operational protocol of the item (2.1.2), we obtain: [15.1]

$$\frac{\partial \varrho}{\partial t} + \frac{\partial (\varrho v_{qu})}{\partial x} = - \frac{\varrho}{2\, \sigma} (\frac{[x - q(t)]}{a^2(t)} - 1\,), \quad (2.3.2.1)$$

expression that indicates decoherence of the considered physical system represented by the Diósi-Halliwell-Nassar Equation (*DHN−E*) [eq. (2.3.2)]; then the *Continuity Equation* its not preserved.

Now, let us obtained another dynamic equation of the *DHN−E*. So, differentiating the eq. (2.3.1.3) with respect x and using the eqs. (2.1.2.1a-e), we obtain:

$$- \hbar \frac{\partial^2 S}{\partial x\, \partial t} = - \frac{\hbar^2}{2m} \frac{\partial}{\partial x} \left[\frac{1}{\varphi} \frac{\partial^2 \varphi}{\partial x^2} - \left(\frac{\partial S}{\partial x} \right)^2 \right] + \frac{\partial V(x, t)}{\partial x} + \lambda\, X(t) \ \rightarrow$$

$$\frac{\partial}{\partial t} \left(\frac{\hbar}{m} \frac{\partial S}{\partial x} \right) =$$

$$= - \frac{\partial}{\partial x} \left(- \frac{\hbar^2}{2\, m^2} \frac{1}{\varphi} \frac{\partial^2 \varphi}{\partial x^2} \right) - \frac{1}{2} \frac{\partial}{\partial x} \left(\frac{\hbar}{m} \frac{\partial S}{\partial x} \right)^2 - \frac{1}{m} \frac{\partial V(x, t)}{\partial x} - \frac{\lambda}{m} X(t) \ \rightarrow$$

$$\frac{\partial v_{qu}}{\partial t} + v_{qu} \frac{\partial v_{qu}}{\partial x} + \frac{\lambda}{m} X(t) = - \frac{1}{m} \frac{\partial}{\partial x} (V + V_{qu}) \,. \qquad (2.3.2.2)$$

We observe that the eq. (2.3.2.2) had the aspect of the *Euler Equation* [8] for a ideal fluid in movement.

Considering the *substantive differentiation* (local plus convective) or *hydrodynamic differentiation*, given by the eqs. (2.1.2.5a,b) and inserting into eq. (2.3.2.2), results:

$$m \frac{d^2 x}{dt^2} = - \frac{\partial}{\partial x} \left[\lambda\, x\, X(t) + V(x, t) + V_{qu}(x, t) \right] \ \rightarrow$$

$$m \frac{d^2 x}{dt^2} = - \lambda\, x + F_C(x, t)|_{x=x(t)} + F_Q(x, t)|_{x=x(t)} \,, \qquad (2.3.2.3)$$

eq. that has the form of the *Second Newton Law*, being the terms of the second member, respectively, the *classical newtonian force* and the *quantum bohmnian force*.

2.3.3 The Quantum Wave Packet of the Linearized Diósi-Halliwell-Nassar Equation along a Classical Trajetory

In order to find the quantum wave packet of the linearized Diósi-Halliwell-Nassar Equation (*DHN−E*) along a classical trajetory, we calculate the v_{qu}. So, using the same operational protocol of the item (2.1.3), we have: [15.1]

$$v_{qu}(x, t) \equiv \frac{dx(t)}{dt} = \frac{\dot{a}(t)}{a(t)} [x - q(t)] + \dot{q}(t). \qquad (2.3.3.1)$$

Multiplying the eq. (2.3.3.1) by $\varrho(x, t)$ [eq. (2.1.3.1a)] and differenting the result in the variable x, we have [remenber that $a(t)$ and $q(t)$]:

$$\varrho(x, t) \times v_{qu}(x, t) = \varrho(x, t) \times \left\{ \frac{\dot{a}(t)}{a(t)} [x - q(t)] + \dot{q}(t) \right\},$$

$$\frac{\partial}{\partial x} (\varrho\, v_{qu}) = \left(\frac{\dot{a}}{a} \right) \times \frac{\partial}{\partial x} [\varrho\, (x - q)] + \dot{q} \frac{\partial \varrho}{\partial x} \ \rightarrow$$

$$\frac{\partial}{\partial x} (\varrho\, v_{qu}) = \left(\frac{\dot{a}}{a} \right) \times \left[(x - q) \frac{\partial \varrho}{\partial x} + \varrho \right] + \dot{q} \frac{\partial \varrho}{\partial x} \ \rightarrow$$

$$\frac{\partial}{\partial x} (\varrho\, v_{qu}) = \varrho \left(\frac{\dot{a}}{a} \times \right) + \frac{\dot{a}}{a} \left[(x - q) \frac{\partial \varrho}{\partial x} \right] + \dot{q} \frac{\partial \varrho}{\partial x} \ \rightarrow$$

$$\frac{\partial}{\partial x}(\varrho\, v_{qu}) = \varrho\,(\frac{\dot{a}}{a}) + \frac{\partial\varrho}{\partial x}[\,(\frac{\dot{a}}{a})\,(x - q) + \dot{q}\,]\,. \quad (2.3.3.2)$$

Using the eq. (2.1.3.1a), we have: [4]

$$\frac{\partial\varrho}{\partial x} = -\,\varrho \times \frac{(x-q)}{a^2}\,. \quad (2.3.3.3)$$

Inserting the eq. (2.3.3.3) into eq. (2.3.3.2), results:

$$\frac{\partial}{\partial x}(\varrho\, v_{qu}) = \varrho\,(\frac{\dot{a}}{a}) - \varrho \times \frac{(x-q)}{a^2} \times [\,(\frac{\dot{a}}{a})\,(x - q) + \dot{q}\,. \quad (2.3.3.4)$$

Additing the eqs. (2.1.3.2) and (2.3.3.4), we have:

$$\frac{\partial\varrho}{\partial t} + \frac{\partial}{\partial x}(\varrho\, v_{qu}) = \varrho \times \{-\,\frac{\dot{a}}{a} + \frac{\dot{q}}{a^2}(x-q) + \frac{\dot{a}}{a^3}(x-q)^2\,\} +$$

$$+\,\varrho \times \{\,(\frac{\dot{a}}{a}) - \frac{(x-q)}{a^2} \times [\,(\frac{\dot{a}}{a})\,(x-q)\,\dot{q}\,]\,\} = 0 \;\;\rightarrow$$

$$\frac{\partial\varrho}{\partial t} + \frac{\partial}{\partial x}(\varrho\, v_{qu}) = 0, \quad (2.3.3.5)$$

result that is incompatible with the eq. (2.3.2.1). So, to make compatible it is necessary to substitute the eq. (2.3.3.1) by one *modified quantum velocity* (v_{qum}) defined by: [14]

$$v_{qum} = [\,\frac{\dot{a}(t)}{a(t)} + \frac{1}{2\,\sigma}\,] \times [x - q(t)] + \dot{q}(t). \quad (2.3.3.6)$$

We observe that the integration of the eq. (2.3.3.6) give us the *bohmnian quantum trajectory* of the physical system represented by *DHN−E*.

To obtain the quantum wave packet [$\Psi(x\,,\,t)$] of the *DHN−E* given by eq. (2.3.2), let us expand the functions S(x, t), V(x t), and V_{qu}(x, t) around of $q(t) = \langle x \rangle$ up to *second Taylor order*. [9] In this way, using the eq. (2.1.3.14) we have:

$$S(x,t) = S[q(t),t] + S'[q(t),t] \times [x - q(t)] + \frac{S''[\mathrm{q(t)\ t}]}{2} \times [\mathrm{x - q(t)}]^2, \quad (2.3.3.7)$$

$$V(x,t) = V[q(t),t] + V'[q(t),t] \times [x - q(t)] + \frac{V''[\mathrm{q(t),\ t}]}{2} \times [\mathrm{x - q(t)}]^2, \quad (2.3.3.8)$$

$$V_{qu}(x,t) = V_{qu}[q(t),t] + V'_{qu}[q(t),t] \times [x - q(t)] +$$

$$+\,\frac{1}{2}\,V''_{\mathrm{qu}}[\mathrm{q(t),\ t}] \times [\mathrm{x - q(t)}]^2, \quad (2.3.3.9a)$$

$$V_{qu}(x,t) = \frac{\hbar^2}{4\,m\,a^2(t)} - \frac{\hbar^2}{8\,m\,a^4(t)} \times [x - q(t)]^2\,. \quad (2.3.3.9b)$$

Differenting the eq. (2.3.3.7) in the variable *x*, multiplying the result by ħ/*m* and using the eqs. (2.1.2.1c,d) and (2.3.3.6), we have:

$$\frac{\hbar}{m}\frac{\partial S(x,t)}{\partial t} = \frac{\hbar}{m}\{ S'[q(t),t] + S''[q(t),t] \times [x - q(t)] \} = v_{qum}(x,t) =$$

$$= [\frac{\dot{a}(t)}{a(t)} + \frac{1}{2\sigma}] \times [x - q(t)] + \dot{q}(t) \rightarrow$$

$$S'[q(t),t] = \frac{m\dot{q}(t)}{\hbar}, \quad S''[q(t),t] = \frac{\hbar}{m}[\frac{\dot{a}(t)}{a(t)} + \frac{1}{2\sigma}]. \quad (2.3.3.10a,b)$$

Substituting the eqs. (2.3.3.10a,b) into eq. (2.3.3.7), results:

$$S(x,t) = S_0(t) + \frac{m\dot{q}(t)}{\hbar} \times [x - q(t)] + \frac{m}{2\hbar}[\frac{\dot{a}(t)}{a(t)} + \frac{1}{2\sigma}] \times [x - q(t)]^2, \quad (2.3.3.11a)$$

where [see eq. (2.1.4.11b)]:

$$S_o(t) \equiv S[q(t),t]. \quad (2.3.3.11b)$$

Differenting the eq. (2.3.3.11a) in relation to the time t and using the eq. (2.1.3.2), results (remember that $\partial x/\partial t = 0$):

$$\frac{\partial S(x,t)}{\partial t} = \dot{S}_o(t) + \frac{m}{\hbar}\frac{\partial}{\partial t}\{\dot{q}(t)[x - q(t)]\} + \frac{m}{2\hbar}\frac{\partial}{\partial t}\{[\frac{\dot{a}(t)}{a(t)} + \frac{1}{2\sigma}] \times [x - q(t)]^2\} =$$

$$= \dot{S}_o(t) + \frac{m}{\hbar}\{\ddot{q}(t)[x - q(t)] - \dot{q}^2(t)\} +$$

$$+ \frac{m}{2\hbar}[\frac{\ddot{a}(t)}{a(t)} - \frac{\dot{a}^2(t)}{a^2(t)}] \times [x - q(t)]^2 - \frac{m\dot{q}(t)}{\hbar}[\frac{\dot{a}(t)}{a(t)} + \frac{1}{2\sigma}] \times [x - q(t)] \rightarrow$$

$$\frac{\partial S(x,t)}{\partial t} = \dot{S}_o(t) - \frac{m}{\hbar}\dot{q}^2 + \frac{m}{\hbar}\{\ddot{q}(t) - \dot{q}(t) \times [\frac{\dot{a}(t)}{a(t)} + \frac{1}{2\sigma}]\} \times [x - q(t)] +$$

$$+ \frac{m}{2\hbar}[\frac{\ddot{a}(t)}{a(t)} - \frac{\dot{a}^2(t)}{a^2(t)}] \times [x - q(t)]^2. \quad (2.3.3.12)$$

Using the eqs. (2.1.2.1c-e) and (2.3.1.3), we obtain:

$$-\hbar\frac{\partial S}{\partial t} = [-\frac{\hbar^2}{2m}\frac{1}{\varphi}\frac{\partial^2\varphi}{\partial x^2} + \frac{m}{2}(\frac{\hbar}{m}\frac{\partial S}{\partial x})^2] + V(x,t) + \lambda\, x\, X(t) \rightarrow$$

$$\hbar\frac{\partial S}{\partial t} + \frac{m}{2}v_{qum}^2 + V(x,t) + V_{qu}(x,t) + \lambda\, x\, X(t) = 0. \quad (2.3.3.13)$$

Inserting the eqs. (2.3.3.6,8,9b) into eq. (2.3.3.13), ordering the result in potencies of $[x, q(t)]$, and considering that $(x - q)^o = 1$, we have: [15.1]

$$\{\hbar\dot{S}_o(t) - \frac{m}{2}\dot{q}^2(t) + V[q(t),t] + \lambda\, q(t)\, X(t) + \frac{\hbar^2}{4\, m\, a^2}\} \times [x - q(t)]^0 +$$

$$+ \{m\ddot{q}(t) + V'[q(t),t] + \lambda\, X(t)\} \times [x - q(t)] +$$

$$+ \{\frac{m}{2}[\frac{\ddot{a}(t)}{a(t)} + \frac{1}{\sigma}\frac{\dot{a}(t)}{a(t)} + \frac{1}{4\sigma^2}] +$$

$$+ \ \frac{1}{2} V''[q(t), t] - \frac{\hbar^2}{8\ m\ a^4(t)} \} \times [x - (q)t)]^2 = 0 \ . \qquad (2.3.3.14)$$

As the above eq. (2.3.3.14)] is an identically null polynomium, the coefficients of the potencies must be all equal to zero, that is:

$$\dot{S}_o(t) = \frac{1}{\hbar} \{ \ \frac{m}{2} \dot{q}^2 - V[q(t), t] - \lambda \ q(t) \ X(t) - \frac{\hbar^2}{4\ m\ a^2(t)} \} \ , \qquad (2.3.3.15)$$

$$\ddot{q} + \frac{1}{m} V'[q(t), t] + \frac{\lambda}{m} X(t) = 0 \ , \qquad (2.3.3.16)$$

$$\ddot{a}(t) + \frac{\dot{a}(t)}{\sigma} + \{ \ \frac{1}{m} V''[q(t), t] + \frac{1}{4\ \sigma^2} \} \times a(t) = \frac{\hbar^2}{4\ m^2\ a^3(t)} \ . \qquad (2.3.3.17)$$

Assuming that the following initial conditions are obeyed [see eqs. (2.1.3.21a-d):

$$q(0) = x_o \ , \quad \dot{q}(0) = v_o \ , \quad a(0) = a_o \ , \quad \dot{a}(0) = b_o \ , \qquad (2.3.3.18a\text{-}d)$$

and that [see eqs.(2.1.2.1c,d) and (2.3.3.11b)]:

$$S_o(0) = \frac{m\ v_o\ x_o}{\hbar} \ , \qquad (2.2.3.19)$$

the integration of the expression (2.3.3.15) will be given by:

$$S_o(t) = \frac{1}{\hbar} \int_o^t dt' \ \{ \ \frac{m}{2} \dot{q}^2(t') - V[q(t'), t'] - \lambda \ q(t') \ X(t') - \frac{\hbar^2}{4\ m\ a^2(t')} \} \ +$$

$$+ \ \frac{m\ v_o\ x_o}{\hbar} \ . \qquad (2.3.3.20)$$

Taking into account the eq. (2.3.3.20) in the eq. (2.3.3.11a) and considereing the eq. (2.3.3.11b), results:

$$S(x, t) = \frac{1}{\hbar} \int_o^t dt' \ \{ \ \frac{m}{2} \dot{q}^2(t') - V[q(t'), t'] - \lambda \ q(t') \ X(t') - \frac{\hbar^2}{4\ m\ a^2(t')} \} + \frac{m\ v_o\ x_o}{\hbar} \ +$$

$$+ \ \frac{m\ \dot{q}(t)}{\hbar} \times [x - q(t)] + \frac{m}{2\ \hbar} [\ \frac{\dot{a}(t)}{a(t)} + \frac{1}{2\ \sigma}] \times [x - q(t)]^2 \ . \qquad (2.3.3.21)$$

The eq. (2.3.3.21) permit us, finally, to obtain the wave packet for the *DHN−E*. Indeed, considering the eqs. (2.1.1.1), (2.1.3.1b) and (2.3.3.21), we get: [15.1]

$$\Psi(x, t) = [2\ \pi\ a^2(t)]^{-1/4} \times exp \ (\ \{ \ \frac{i\ m}{2\ \hbar} [\ \frac{\dot{a}(t)}{a(t)} + \frac{1}{2\ \sigma}] - \frac{1}{4\ a^2(t)} \} \times [x - q(t)]^2) \ \times$$

$$\times \ exp \ \{ \ \frac{i\ m\ \dot{q}(t)}{\hbar} \times [x - q(t)] + \frac{i\ m\ v_o\ x_o}{\hbar} \} \ \times$$

$$\times \ exp \ [\ \frac{i}{\hbar} \int_o^t dt' \ \{ \ \frac{m}{2} m \dot{q}^2(t') - V[q(t'), t'] - \lambda \ q(t') \ X(t') - \frac{\hbar^2}{4\ m\ a^2(t')} \} \ ] . \qquad (2.3.3.22)$$

2.3.4. Calculation of the Feynman Propagator of the Linearized Diósi-Halliwell-Nassar Equation along a Classical Trajetory

For to calculated the Feynman-de Broglie-Bohm propagator of the linearized *DHN−E* along a classical trajectory, will be following the same operational protocol of the itens (2.1.4; 2.2.4). Thus, by using the eqs. (2.1.4.1-3,5,9), we can write that: [15.1]

$$\Psi(x, t) = \int_{-\infty}^{+\infty} K(x, x_o; t, 0)\, \Psi(x_o, 0)\, dx_o\,, \quad (2.3.4.1)$$

$$\Phi(v_o, x, t) = (2\,\pi\, a_o^2)^{1/4}\, \Psi(v_o, x, t)\,, \quad (2.3.4.2)$$

$$\int_{-\infty}^{+\infty} dv_o\, \Phi^*(v_o, x', t)\, \Phi(v_o, x', t) = \left(\frac{2\,\pi\,\hbar}{m}\right) \delta(x - x')\,, \quad (2.3.4.3)$$

$$\varrho(v_o, x, t) = (2\,\pi\, a_o^2)^{-1/4}\, \Phi^*(v_o, x, t)\, \Psi(v_o, x, t)\,, \quad (2.3.4.4)$$

$$\Psi(x, t) = \int_{-\infty}^{+\infty} \left\{ \left(\frac{m}{2\,\pi\,\hbar}\right) \int_{-\infty}^{+\infty} dv_o\, \Phi(v_o, x, t)\, \Phi^*(v_o, x_o, 0)\, \Psi(x_o, 0)\, \Psi(x_o, 0) \right\} \times$$

$$\times\, \Psi(x_o, 0)\, dx_o\,. \quad (2.3.4.5)$$

Comparing the eqs. (2.3.4.1,5), we have:

$$K(x, x_o, t) = \left(\frac{m}{2\,\pi\,\hbar}\right) \int_{-\infty}^{+\infty} dv_0\, \Phi(v_o, x, t)\, \Phi^*(v_o, x_o, 0)\,. \quad (2.3.4.6)$$

Substituting the eq. (2.3.4.2) in the eq. (2.3.4.6), we obtain the Feynman Propagator of the linearized Diósi-Halliwell-Nassar Equation (*DHN−E*) along a classical trajetory, that we were looking for, that is [remembering that $\Phi^*(v_o, x_o, 0) = exp\,(-\frac{i\, m\, v_o\, x_o}{\hbar})$]:

$$K(x, x_o; t) = \left(\frac{m}{2\,\pi\,\hbar}\right) \int_{-\infty}^{+\infty} dv_o \sqrt{\frac{a_0}{a(t)}} \times$$

$$\times\, exp\left(\left\{ \frac{i}{2}\, \frac{m}{\hbar} \left[\frac{\dot{a}(t)}{a(t)} + \frac{1}{2\,\sigma} \right] - \frac{1}{4\, a^2(t)} \right\} \times [x - q(t)]^2 \right\} \times$$

$$\times\, exp\left\{ \frac{i\, m\, \dot{q}(t)}{\hbar} \times [x - q(t)] \right\} \times$$

$$\times\, exp\left[\frac{i}{\hbar} \int_o^t dt' \left\{ \frac{m}{2} \dot{q}^2(t') - V[q(t'), t'] - \lambda\, q(t')\, X(t') - \frac{\hbar^2}{4\, m\, a^2(t')} \right\} \right], \quad (2.3.4.7)$$

where $q(t)$ and $a(t)$ are solutions of the differential equations given by the eqs.(2.3.3.16,17).

2.4. The Kostin Equation

In 1972, M. D. Kostin [16] proposed a non-linear Schrödinger equation, to represent time dependent physical systems, defined by:

$$i\,\hbar\, \frac{\partial \Psi(x, t)}{\partial t} = -\, \frac{\hbar^2}{2\, m}\, \frac{\partial^2\, \Psi(x, t)}{\partial x^2} +$$

$$+ \left[V(x, t) + \frac{\hbar \nu}{2 i} \ell n \frac{\Psi(x, t)}{\Psi^*(x, t)} \right] \times \Psi(x, t) , \quad (2.4.1)$$

where $\Psi(x, t)$ and $V(x, t)$ are, respectively, the wavefunction and the time dependent potential of the physical system in study, and ν is a constant.

2.4.1. The Wave Function of the Kostin Equation

Putting the eqs. (2.1.1.1) and (2.1.1.2a,b) into the eq. (2.4.1), we have: [4,15.5]

$$i \hbar \left(\frac{\partial \varphi}{\partial t} + i \varphi \frac{\partial S}{\partial t} \right) = - \frac{\hbar^2}{2 m} \left[\frac{\partial^2 \varphi}{\partial x^2} + 2 i \frac{\partial S}{\partial x} \frac{\partial \varphi}{\partial x} + i \varphi \frac{\partial^2 S}{\partial x^2} - \varphi \left(\frac{\partial S}{\partial x} \right)^2 \right] +$$

$$+ \left[V(x, t) + \hbar \nu S(x, t) \right] \times \varphi(x, t) , \quad (2.4.1.1)$$

Separating the real and imaginary parts of the relation (2.4.1.1), results:

a) imaginary part

$$\frac{\partial \varphi}{\partial t} = - \frac{\hbar}{2 m} \left(2 \frac{\partial S}{\partial x} \frac{\partial \varphi}{\partial x} + \varphi \frac{\partial^2 S}{\partial x^2} \right) , \quad (2.4.1.2)$$

b) real part

$$- \hbar \frac{\partial S}{\partial t} = - \frac{\hbar^2}{2 m} \left[\frac{1}{\varphi} \frac{\partial^2 \varphi}{\partial x^2} - \varphi \left(\frac{\partial S}{\partial x} \right)^2 \right] + V(x, t) + \lambda \nu S . \quad (2.4.1.3)$$

2.4.2. Dynamics of the Kostin Equation

Now, let us see the correlation between the expressions (2.4.1.2-3) and the traditional equations of the Ideal Fluid Dynamics: [8] a) *Continuity Equation*, b) *Euler's equation* (for conservative systems) or b') *Navier-Stokes equation* (for non-conservative systems). Thus, putting the eqs. (2.1.2.1a-e) into the eq. (2.4.1.2), and using the same operational protocol of the item (2.3.2), we obtain: [4,15.5]

$$\frac{\partial \varrho}{\partial t} + \frac{\partial (\varrho v_{qu})}{\partial x} = 0, \quad (2.4.2.1)$$

expression that indicates coherence of the considered physical system represented by the Kostin Equation ($K{-}E$) [eq. (2.4.1)]; then the *Continuity Equation* it is preserved.

Now, let us obtained another dynamic equation of the $K{-}E$. So, differentiating the eq. (2.4.1.3) with respect x and using the eqs. (2.1.2.1a-e), we obtain:

$$\frac{\partial v_{qu}}{\partial t} + v_{qu} \frac{\partial v_{qu}}{\partial x} + \nu v_{qu} = - \frac{1}{m} \frac{\partial}{\partial x} (V + V_{qu}) . \quad (2.4.2.2)$$

We observe that the eq. (2.4.2.2) had the aspect of the *Euler Equation* [8] for a real fluid in movement.

Considering the *substantive differentiation* (local plus convective) or *hydrodynamic differentiation*, given by the eqs. (2.1.2.5a,b) and inserting into eq. (2.4.2.2), results:

$$m \frac{d^2 x}{dt^2} + \nu \frac{dx}{dt} = - \frac{\partial}{\partial x} \left[V(x, t) + V_{qu}(x, t) \right], \quad (2.4.2.3)$$

what has a form of the *Dissipative Second Newton Law*, being the terms of the second member, respectively, the *classical newtonian force* and the *quantum bohmnian force*.

2.4.3 The Quantum Wave Packet of the Linearized Kostin Equation along a Classical Trajetory

In order to find the quantum wave packet of the linearized Kostin Equation ($K-E$) along a classical trajetory, we calculate the v_{qu}. So, using the same operational protocol of the item (2.3.3), we have: [4,15.5]

$$v_{qu}(x, t) \equiv \frac{dx(t)}{dt} = \frac{\dot{a}(t)}{a(t)} [x - q(t)] + \dot{q}(t). \qquad (2.4.3.1)$$

We observe that the integration of the eq. (2.4.3.1) give us the *bohmnian quantum trajectory* of the physical system represented by $K-E$.

To obtain the quantum wave packet [$\Psi(x, t)$] of the $K-E$ given by eq. (2.4.1), let us expand the functions S(x, t), V(x t), and V_{qu}(x, t) around of $q(t) = <x>$ up to *second Taylor order*. [9] In this way, using the eq. (2.1.3.14) we have:

$$S(x, t) = S[q(t), t] + S'[q(t), t] \times [x - q(t)] + \frac{1}{2} S''[q(t)\ t] \times [x - q(t)]^2, \qquad (2.4.3.2)$$

$$V(x, t) = V[q(t), t] + V'[q(t), t] \times [x - q(t)] +$$

$$+ \frac{1}{2} V''[q(t), t] \times [x - q(t)]^2, \qquad (2.4.3.3)$$

$$V_{qu}(x, t) = V_{qu}[q(t), t] + V'_{qu}[q(t), t] \times [x - q(t)] +$$

$$+ \frac{1}{2} V''_{qu}[q(t), t] \times [x - q(t)]^2, \qquad (2.4.3.4a)$$

$$V_{qu}(x, t) = \frac{\hbar^2}{4\, m\, a^2(t)} - \frac{\hbar^2}{8\, m\, a^4(t)} \times [x - q(t)]^2 . \qquad (2.4.3.4b)$$

Differenting the eq. (2.4.3.2) in the variable x, multiplying the result by $\hbar/m$ and using the eqs. (2.1.2.1c,d) and (2.4.3.1), we have:

$$\frac{\hbar}{m} \frac{\partial S(x, t)}{\partial t} = \frac{\hbar}{m} \{ S'[q(t), t] + S''[q(t), t] \times [x - q(t)] \} = v_{qu}(x, t) =$$

$$= [\frac{\dot{a}(t)}{a(t)}] \times [x - q(t)] + \dot{q}(t) \quad \rightarrow$$

$$S'[q(t), t] = \frac{m\,\dot{q}(t)}{\hbar}, \quad S''[q(t), t] = \frac{\hbar}{m} [\frac{\dot{a}(t)}{a(t)}] . \qquad (2.4.3.5a,b)$$

Substituting the eqs. (2.4.3.5a,b) into eq. (2.4.3.2), results:

$$S(x, t) = S_0(t) + \frac{m\,\dot{q}(t)}{\hbar} \times [x - q(t)] + \frac{m}{2\,\hbar} [\frac{\dot{a}(t)}{a(t)}] \times [x - q(t)]^2 , \qquad (2.4.3.6a)$$

where [see eq. (2.3.3.11b)]:

$$S_o(t) \equiv S[q(t), t] . \qquad (2.4.3.6b)$$

Differenting the eq. (2.4.3.6a) in relation to the time t and using the eq. (2.1.3.2), results (remember that $\partial x/\partial t = 0$): [15.5]

$$\frac{\partial S(x,t)}{\partial t} = \dot{S}_o(t) - \frac{m\,\dot{q}^2(t)}{\hbar} + \frac{m}{\hbar}\,[\ddot{q}(t) - \frac{\dot{q}(t)\,\dot{a}(t)}{a(t)}\times[x - q(t)] +$$

$$+ \frac{m}{2\,\hbar}\,[\frac{\ddot{a}(t)}{a(t)} - \frac{\dot{a}^2(t)}{a^2(t)}] \times [x - q(t)]^2 . \quad (2.4.3.7)$$

Using the eqs. (2.1.2.1c-e) and (2.4.1.3), we obtain:

$$-\hbar\,\frac{\partial S}{\partial t} = [-\frac{\hbar^2}{2\,m}\,\frac{1}{\varphi}\,\frac{\partial^2\varphi}{\partial x^2} + \frac{m}{2}\,(\frac{\hbar}{m}\,\frac{\partial S}{\partial x})^2] + V(x\,,t) + \hbar\,\nu\,S \;\;\rightarrow$$

$$\hbar\,\frac{\partial S}{\partial t} + \frac{m}{2}\,v_{qu}^2 + V(x,t) + V_{qu}(x,t) + \hbar\,\nu\,S = 0 . \quad (2.4.3.8)$$

Inserting the eqs. (2.4.3.1,3,4b) into eq. (2.4.3.8), ordering the result in potencies of $[x\,,\,q(t)]$, and considering that $(x - q)^o = 1$, we have: [15.5]

$$\{\hbar\,\dot{S}_o(t) - \frac{m}{2}\dot{q}^2(t) + V[q(t),t] + \hbar\,\nu\,S_0(t) + \frac{\hbar^2}{4\,m\,a^2(t)}\} \times [x - q(t)]^0 +$$

$$+ \{m\,\ddot{q}(t) + V'[q(t),t] + m\,\nu\,\dot{q}(t)\} \times [x - q(t)] +$$

$$+ \{\frac{m}{2}\,[\frac{\ddot{a}(t)}{a(t)}] + m\,\nu\,\frac{\dot{a}(t)}{a(t)} + \frac{1}{2}\,V''[\mathrm{q(t), t}] - \frac{\hbar^2}{8\,\mathrm{m}\,\mathrm{a}^4(\mathrm{t})}\} \times [\mathrm{x - (q)t)}]^2 = 0 . \quad (2.4.3.9)$$

As the above relation [eq. (2.4.3.9)] is an identically null polynomium, the coefficients of the potencies must be all equal to zero, that is:

$$\dot{S}_o(t) = \frac{1}{\hbar}\,\{\frac{m}{2}\dot{q}^2(t) - V[q(t),t] - \hbar\,\nu\,S_0(t) - \frac{\hbar^2}{4\,m\,a^2(t)}\}\,, \quad (2.4.3.10)$$

$$\ddot{q}(t) + \nu\,\dot{q}(t) + \frac{1}{m}\,V'[q(t),t] = 0\,, \quad (2.4.3.11)$$

$$\ddot{a}(t) + \nu\,\dot{a}(t) + \frac{1}{m}\,V''[\mathrm{q(t), t}] \times \mathrm{a(t)} = \frac{\hbar^2}{4\,\mathrm{m}^2\,\mathrm{a}^3(\mathrm{t})} . \quad (2.4.3.12)$$

Assuming that the following initial conditions are obeyed [see eqs. (2.1.3.21a-d):

$$q(0) = x_o\,,\quad \dot{q}(0) = v_o\,,\quad a(0) = a_o\,,\quad \dot{a}(0) = b_o\,, \quad (2.4.3.13\text{a-d})$$

and that [see eqs. (2.1.2.1c-d) and (2.4.3.6b)]:

$$S_o(0) = \frac{m\,v_o\,x_o}{\hbar}\,, \quad (2.4.3.14)$$

the integration of the expression (2.4.3.10) will be given by:

$$S_o(t) = \frac{1}{\hbar}\int_o^t dt'\,\{\frac{m}{2}\dot{q}^2(t') - V[q(t'),t'] - \hbar\,\nu\,S_0(t') - \frac{\hbar^2}{4\,m\,a^2(t')}\} +$$

$$+\ \frac{m\, v_o\, x_o}{\hbar}\ . \qquad (2.4.3.15)$$

Taking into account the eq. (2.4.3.15) in the eq. (2.4.3.6a) and considering the eq. (2.4.3.6b), results:

$$S(x,\, t) = \frac{1}{\hbar} \int_o^t dt' \ \{ \ \frac{m}{2} \dot{q}^2(t') - \nu\, S_0(t') - V[q(t'),\, t'] - \frac{\hbar^2}{4\, m\, a^2(t')} \} + \frac{m\, v_o\, x_o}{\hbar} +$$

$$+\ \frac{m\, \dot{q}(t)}{\hbar} \times [x - q(t)] + \frac{m}{2\, \hbar} [\ \frac{\dot{a}(t)}{a(t)} \ ] \times [x - q(t)]^2\ . \qquad (2.4.3.16)$$

The eq. (2.4.3.16) permit us, finally, to obtain the wave packet for the $K{-}E$. Indeed, considering the eqs. (2.1.1.1), (2.1.3.1b) and (2.4.3.15), we get: [15.5]

$$\Psi(x,\, t) = [2\, \pi\, a^2(t)]^{-\,1/4} \times exp\ (\ \{\ \frac{i\, m}{2\, \hbar} [\ \frac{\dot{a}(t)}{a(t)} \ ] - \frac{1}{4\, a^2(t)} \} \times [x - q(t)]^2\ ) \times$$

$$\times\ exp\ \{\ \frac{i\, m\, \dot{q}(t)}{\hbar} \times [x - q(t)] + \frac{i\, m\, v_o\, x_o}{\hbar} \} \times$$

$$\times\ exp\ [\ \frac{i}{\hbar} \int_o^t dt' \ \{\ \frac{m}{2}\, m\, \dot{q}^2(t') - V[q(t'),\, t'] - \hbar\, \nu\, S_0(t') - \frac{\hbar^2}{4\, m\, a^2(t')} \} \ ]. \qquad (2.4.3.17)$$

2.4.4. Calculation of the Feynman Propagator of the Linearized Kostin Equation along a Classical Trajetory

For to calculated the Feynman-de Broglie-Bohm propagator of the linearized $K{-}E$ along a classical trajectory, will be following the same operational protocol of the item (2.3.4). Thus, by using the eqs. (2.1.4.1-3,5,9), we can write that: [15.5]

$$\Psi(x,\, t) = \int_{-\infty}^{+\infty} K(x,\, x_o;t,\, 0)\ \Psi(x_o,\, 0)\ dx_o\ , \qquad (2.4.4.1)$$

$$\Phi(v_o,\, x,\, t) = (2\, \pi\, a_o^2)^{1/4}\ \Psi(v_o,\, x,\, t)\ , \qquad (2.4.4.2)$$

$$\int_{-\infty}^{+\infty} dv_o\ \Phi^*(v_o,\, x',\, t)\ \Phi(v_o,\, x',\, t) = (\ \frac{2\, \pi\, \hbar}{m}\ )\ \delta(x - x')\ , \qquad (2.4.4.3)$$

$$\varrho(v_o,\, x,\, t) = (2\, \pi\, a_o^2)^{1/4}\ \Phi^*(v_o,\, x,\, t)\ \Psi(v_o,\, x,\, t)\ , \qquad (2.4.4.4)$$

$$\Psi(x,\, t) = \int_{-\infty}^{+\infty} \{\ (\ \frac{m}{2\, \pi\, \hbar}\ ) \int_{-\infty}^{+\infty} dv_o\ \Phi(v_o,\, x,\, t)\ \Phi^*(v_o,\, x_o,\, 0)\ \} \times$$

$$\times\ \Psi(x_o,\, 0)\ dx_o\ . \qquad (2.4.4.5)$$

Comparing the eqs. (2.4.4.1,5), we have:

$$K(x,\, x_o,\, t) = (\ \frac{m}{2\, \pi\, \hbar}\ ) \int_{-\infty}^{+\infty} dv_0\ \Phi(v_o,\, x,\, t)\ \Phi^*(v_o,\, x_o,\, 0)\ . \qquad (2.4.4.6)$$

Substituting the eqs.(2.4.3.17) and (2.4.4.2) in the eq. (2.4.4.6), we obtain the Feynman Propagator of the linearized Kostin Equation along a classical trajetory, that we were looking for, that is [remembering that $\Phi^*(v_o, x_o, 0) = exp\,(-\frac{i\,m\,v_o\,x_o}{\hbar})$]:

$$K(x, x_o; t) = \left(\frac{m}{2\,\pi\,\hbar}\right) \int_{-\infty}^{+\infty} dv_o \sqrt{\frac{a_0}{a(t)}} \times$$

$$\times\, exp\left(\left\{ \frac{i}{2}\,\frac{m}{\hbar}\left[\frac{\dot{a}(t)}{a(t)}\right] - \frac{1}{4\,a^2(t)} \right\} \times [x - q(t)]^2 \right) \times$$

$$\times\, exp\left\{ \frac{i\,m\,\dot{q}(t)}{\hbar} \times [x - q(t)] \right\} \times$$

$$\times\, exp\left[\frac{i}{\hbar} \int_o^t dt'\left\{ \frac{m}{2}\dot{q}^2(t') - V[q(t'), t'] - \hbar\,\nu\, S_o(t') - \frac{\hbar^2}{4\,m\,a^2(t')} \right\} \right], \quad (2.4.4.7)$$

where $q(t)$ and $a(t)$ are solutions of the differential equations given by the eqs.(2.4.3.11,12).

Finally, it is important to note that putting $\lambda = 0$ and $V[q(t'), t'] = 0$ into eqs. (2.4.3.11,12) and (2.4.4.7), we obtain the free Feynman propagator. [2]

2.5. The Schuch-Chung-Hartmann Equation

In 1983-1985, D. Schuch, K. M. Chung and Hermann Hartmann [17] proposed a non-linear Schrödinger equation, to represent time dependent physical systems, defined by:

$$i\,\hbar\,\frac{\partial\Psi(x, t)}{\partial t} = -\frac{\hbar^2}{2\,m}\,\frac{\partial^2\,\Psi(x, t)}{\partial x^2} +$$

$$+ \left\{ V(x, t) + \frac{\hbar\,\nu}{i}\left[\ell n\Psi(x, t) - \langle \ell n\Psi(x, t)\rangle \right] \right\} \times \Psi(x, t)\,, \quad (2.5.1)$$

where $\Psi(x, t)$ and $V(x, t)$ are, respectively, the wavefunction and the time dependent potential of the physical system in study, and ν is a constant.

2.5.1. The Wave Function of the Schuch-Chung-Hartmann Equation

Putting the eqs. (2.1.1.1) and (2.1.1.2a,b) into the eq. (2.5.1), we have: [4,15.6]

$$i\,\hbar\left(\frac{\partial\varphi}{\partial t} + i\,\varphi\,\frac{\partial S}{\partial t}\right) = -\frac{\hbar^2}{2\,m}\left[\frac{\partial^2\varphi}{\partial x^2} + 2\,i\,\frac{\partial S}{\partial x}\,\frac{\partial\varphi}{\partial x} + i\,\varphi\,\frac{\partial^2 S}{\partial x^2} - \varphi\left(\frac{\partial S}{\partial x}\right)^2\right] +$$

$$+ \left\{ V(x, t) - i\,\hbar\,\nu\left[\ell n\varphi - \langle \ell n\varphi\rangle + i(S - \langle S\rangle) \right] \right\} \times \varphi(x, t)\,, \quad (2.5.1.1)$$

Separating the real and imaginary parts of the relation (2.5.1.1), results:

a) <u>imaginary part</u>

$$\frac{\partial\varphi}{\partial t} = -\frac{\hbar}{2\,m}\left(2\,\frac{\partial S}{\partial x}\,\frac{\partial\varphi}{\partial x} + \varphi\,\frac{\partial^2 S}{\partial x^2}\right) - \nu\,(\ell n\varphi - \langle \ell n\varphi\rangle)\,, \quad (2.5.1.2)$$

b) <u>real part</u>

$$- \hbar \frac{\partial S}{\partial t} = - \frac{\hbar^2}{2\,m} [\frac{1}{\varphi} \frac{\partial^2 \varphi}{\partial x^2} - (\frac{\partial S}{\partial x})^2] + V(x, t) + \hbar\, \nu\, (S - <S>) . \quad (2.5.1.3)$$

2.5.2. Dynamics of the Schuch-Chung-Hartmann Equation

Now, let us see the correlation between the expressions (2.5.1.2-3) and the traditional equations of the Ideal Fluid Dynamics: [8] a) *Continuity Equation*, b) *Euler's equation* (for conservative systems) or b') *Navier-Stokes equation* (for non-conservative systems). Thus, putting the eqs. (2.1.2.1a-f) into the eq. (2.5.1.2), and using the same operational protocol of the item (2.4.2), we obtain: [4,15.6]

$$\frac{\partial \varrho}{\partial t} + \frac{\partial(\varrho v_{qu})}{\partial x} = - \nu\, \varrho\, (\ell n\varrho - <\ell n\varrho>), \quad (2.5.2.1)$$

expression that indicates <u>decoherence</u> of the considered physical system represented by the Schuch-Chung-Hartmann Equation (*SCH−E*) [eq. (2.5.1)]; then the *Continuity Equation* it is not preserved.

Now, let us obtained another dynamic equation of the *SCH−E*. Considering the eqs. (2.1.2.1a-e) and (2.5.1.3), we obtain: [15.6]

$$\hbar\, [\frac{\partial S}{\partial t} + \nu\, (S- - <S>) + \frac{1}{2} m\, v_{qu}^2 + V + V_{qu} = 0 . \quad (2.5.2.2)$$

Considering that:

$$<f(x, t)> = \int_{-\infty}^{+\infty} \varrho(x, t)\, f(x, t)\, dx = g(t) , \quad (2.5.2.3)$$

then:

$$\frac{\partial}{\partial x} <S(x, t)> = \frac{\partial}{\partial x} \int_{-\infty}^{+\infty} \varrho(x, t)\, S(x, t)\, dx = \frac{\partial g(t)}{\partial x} = 0. \quad (2.5.2.4)$$

Now, differentiating the eq. (2.5.1.3) with respect *x,* and using the eqs. (2.1.2.1c-f) and (2.5.2.4), we have: [15.6]

$$\frac{\partial v_{qu}}{\partial t} + v_{qu} \frac{\partial v_{qu}}{\partial x} + \nu\, v_{qu} = - \frac{1}{m} (V + V_{qu}) . \quad (2.5.2.5)$$

We observe that the eq. (2.5.2.5) has the aspect of the *Navier-Stokes Equation* [8] for a real fluid in movement.

Considering the *substantive differentiation* (local plus convective) or *hydrodynamic differentiation*, given by the eqs. (2.1.2.5a,b) and inserting into eq. (2.5.2.5), results:

$$\frac{d^2 x}{dt^2} + \nu \frac{dx}{dt} = - \frac{1}{m} \frac{\partial}{\partial x} [V(x, t) + V_{qu}(x, t)], \quad (2.5.2.6)$$

what has a form of the *Dissipative Second Newton Law*, being the terms of the second member, respectively, the *classical newtonian force* and the *quantum bohmnian force*.

2.5.3 The Quantum Wave Packet of the Linearized Schuch-Chung-Hartmann Equation along a Classical Trajetory

In order to find the quantum wave packet of the linearized Schuch-Chung-Hartmann Equation (*SCH−E*) along a classical trajetory, we must calculate one *modified quantum velocity* (v_{qum}) that satisfating the *Continuity Equation*. So, taking the eq. (2.1.3.1a), can be proven that: [4,15.6]

$$\ell n\varrho - <\ell n\varrho> = - \frac{a^2(t)}{2\, \varrho} \frac{\partial^2 \varrho}{\partial x}. \quad (2.5.3.1)$$

Putting the eq. (2.5.3.1) into eq. (2.5.2.1), results:

$$\frac{\partial \varrho}{\partial t} + \frac{\partial(\varrho v_{qu})}{\partial x} = -\,\nu\,\varrho\,(-\,\frac{a^2}{2\,\varrho}\,\frac{\partial^2 \varrho}{\partial x}\,) = \frac{\nu\,a^2}{2}\,\frac{\partial}{\partial x}\,(\,\frac{\partial \varrho}{\partial x}\,) \;\rightarrow$$

$$\frac{\partial \varrho}{\partial t} + \frac{\partial}{\partial x}\,[\,\varrho\,(\,v_{qu} - \frac{\nu\,a^2}{2\,\varrho}\,\frac{\partial \varrho}{\partial x}\,)\,] = 0. \qquad (2.5.3.2)$$

Defining: [18]

$$v_{qum} = v_{qu} - \frac{\nu}{2}\,\frac{a^2}{\varrho}\,\frac{\partial \varrho}{\partial x}\,, \qquad (2.5.3.3)$$

then the eq. (2.5.3.2) will be the form:

$$\frac{\partial \varrho}{\partial t} + \frac{\partial(\varrho v_{qum})}{\partial x} = 0, \qquad (2.5.3.4)$$

expression that indicates <u>coherence</u> of the *SCH−E*.

So, using the same operational protocol of the item 2.4.3, results: [4,15.6]

$$v_{qum}(x,\,t) \equiv \frac{dx(t)}{dt} = \frac{\dot{a}(t)}{a(t)}\,[x - q(t)] + \dot{q}(t). \qquad (2.5.3.5)$$

Considering the eqs. (2.3.3.3) and (2.5.3.3,5), we have:

$$v_{qu}(x,\,t) \equiv \frac{dx(t)}{dt} = [\,\frac{\dot{a}(t)}{a(t)} - \frac{\nu}{2}\,] \times [x - q(t)] + \dot{q}(t). \qquad (2.5.3.6)$$

We observe that the integration of the eq. (2.5.3.6) give us the *bohmnian quantum trajectory* of the physical system represented by *SCH−E*.

To obtain the quantum wave packet [$\Psi(x\,,\,t)$] of the *SCH−E* given by eq. (2.5.1), let us expand the functions S(x, t), V(xt), and $V_{qu}(x,\,t)$ around of $q(t) = \langle x \rangle$ up to *second Taylor order*. [9] In this way, using the eq. (2.1.3.14) we have:

$$S(x,\,t) = S[q(t),\,t] + S'[q(t),\,t] \times [x - q(t)] + \frac{1}{2}\,S''[q(t)\;t] \times [x - q(t)]^2, \qquad (2.5.3.7)$$

$$V(x,\,t) = V[q(t),\,t] + V'[q(t),\,t] \times [x - q(t)] +$$

$$+\;\frac{1}{2}\,V''[q(t),\,t] \times [x - q(t)]^2, \qquad (2.5.3.8)$$

$$V_{qu}(x,\,t) = V_{qu}[q(t),\,t] + V'_{qu}[q(t),\,t] \times [x - q(t)] +$$

$$+\;\frac{1}{2}\,V''_{qu}[q(t),\,t] \times [x - q(t)]^2, \qquad (2.5.3.9a)$$

$$V_{qu}(x,\,t) = \frac{\hbar^2}{4\,m\,a^2(t)} - \frac{\hbar^2}{8\,m\,a^4(t)} \times [x - q(t)]^2\,. \qquad (2.5.3.9b)$$

Differenting the eq. (2.5.3.7) in the variable *x*, multiplying the result by ħ/*m* and using the eqs. (2.1.2.1c,d) and (2.5.3.6), results:

$$\frac{\hbar}{m}\,\frac{\partial S(x,t)}{\partial x} = \frac{\hbar}{m}\,\{\, S'[q(t), t] + S''[q(t), t] \times [x - q(t)] \,\} = v_{qu}(x, t) =$$

$$= [\, \frac{\dot{a}(t)}{a(t)} - \frac{\nu}{2} \,] \times [x - q(t)] + \dot{q}(t) \;\; \rightarrow$$

$$S'[q(t), t] = \frac{m\,\dot{q}(t)}{\hbar}\,, \quad S''[q(t), t] = \frac{m}{\hbar}\,[\, \frac{\dot{a}(t)}{a(t)} - \frac{\nu}{2} \,]\,. \qquad (2.5.3.10a,b)$$

Substituting the eqs. (2.5.3.10a,b) into eq. (2.5.3.7), we have:

$$S(x, t) = S_0(t) + \frac{m\,\dot{q}(t)}{\hbar} \times [x - q(t)] + \frac{m}{2\,\hbar}\,[\, \frac{\dot{a}(t)}{a(t)} - \frac{\nu}{2} \,] \times [x - q(t)]^2\,, \qquad (2.5.3.11a)$$

where [see eq. (2.4.3.6b)]:

$$S_o(t) \equiv S[q(t), t]\,. \qquad (2.5.3.11b)$$

Now, considering that: [19]

$$\int_{-\infty}^{+\infty} z^n\, exp(-\,z^2)\, dz = \frac{1}{2}\,\sqrt{\pi};\; 0;\; \sqrt{\pi}, (n = 2;\, 1;\, 0) \qquad (2.5.3.12a\text{-}c)$$

and using the eqs. (2.1.3.1a), (2.5.2.3) and (2.5.3.11a), results: [15.6]

$$<S> = S_0(t) + \frac{m}{2\,\hbar}\,[\, \frac{\dot{a}(t)}{a(t)} - \frac{\nu}{2} \,]. \qquad (2.5.3.13)$$

Differenting the eq. (2.5.3.11a) in relation to the time t and using the eq. (2.1.3.2), results (remember that $\partial x/\partial t = 0$): [15.6]

$$\frac{\partial S(x,t)}{\partial t} = \dot{S}_o(t) - \frac{m\,\dot{q}^2(t)}{\hbar} + \frac{m}{\hbar}\,\{\, \ddot{q}(t) - \dot{q}(t)\,[\, \frac{\dot{a}(t)}{a(t)} - \frac{\nu}{2} \,]\,\} \times [x - q(t)] +$$

$$+ \frac{m}{2\,\hbar}\,[\, \frac{\ddot{a}(t)}{a(t)} - \frac{\dot{a}^2(t)}{a^2(t)} \,] \times [x - q(t)]^2\,. \qquad (2.5.3.14)$$

Using the eqs. (2.1.2.1c-e) and (2.5.1.3), we obtain:

$$-\,\hbar\,\frac{\partial S}{\partial t} = [-\,\frac{\hbar^2}{2\,m}\,\frac{1}{\varphi}\,\frac{\partial^2 \varphi}{\partial x^2} + \frac{m}{2}\,(\,\frac{\hbar}{m}\,\frac{\partial S}{\partial x})^2] + V(x\,,t) + \hbar\,\nu\,(S - <S>) \;\; \rightarrow$$

$$\hbar\,\frac{\partial S}{\partial t} + \frac{m}{2}\,v_{qu}^2 + V(x, t) + V_{qu}(x, t) + \hbar\,\nu\,(S - <S>) = 0\,. \qquad (2.5.3.15)$$

Inserting the eqs. (2.5.3.6,8,10a,b) into eq. (2.5.3.15), ordering the result in potencies of [x , $q(t)$], and considering that $(x - q)^o = 1$, we have: [15.6]

$$\{\hbar\,\dot{S}_o(t) - \frac{m}{2}\,\dot{q}^2(t) - \frac{m\,\nu}{2}\,[\, \frac{\dot{a}(t)}{a(t)} - \frac{\nu}{2} \,] + V[q(t), t] + \frac{\hbar^2}{4\,m\,a^2(t)}\,\} \times [x - q(t)]^0 +$$

$$+ \{m\,\ddot{q}(t) + m\,\nu\,\dot{q}(t) + V'[q(t), t]\} \times [x - q(t)] +$$

$$+ \{ \frac{m}{2} \frac{\ddot{a}(t)}{a(t)} - \frac{m \nu^2}{8} + \frac{1}{2} V''[q(t), t] - \frac{\hbar^2}{8 m a^4(t)} \} \times [x - (q)t)]^2 = 0 . \quad (2.5.3.16)$$

As the above relation [eq. (2.5.3.16)] is an identically null polynomium, the coefficients of the potencies must be all equal to zero, that is:

$$\dot{S}_o(t) = \frac{1}{\hbar} \{ \frac{m}{2} \dot{q}^2(t) - V[q(t), t] + \frac{m \nu}{2} [\frac{\dot{a}(t)}{a(t)} - \frac{\nu}{2}] - \frac{\hbar^2}{4 m a^2(t)} \} , \quad (2.5.3.17)$$

$$\ddot{q}(t) + \nu \dot{q}(t) + \frac{1}{m} V'[q(t), t] = 0 , \quad (2.5.3.18)$$

$$\ddot{a}(t) + \nu \dot{a}(t) + \frac{1}{m} \{ V''[q(t), t] - \frac{\nu^2}{4} \} \times a(t) = \frac{\hbar^2}{4 m^2 a^3(t)} . \quad (2.5.3.19)$$

Assuming that the following initial conditions are obeyed [see eqs. (2.1.3.21a-d):

$$q(0) = x_o , \quad \dot{q}(0) = v_o , \quad a(0) = a_o , \quad \dot{a}(0) = b_o , \quad (2.5.3.20a\text{-}d)$$

and that [see eqs. (2.1.2.1c-d) and (2.5.3.11b)]:

$$S_o(0) = \frac{m v_o x_o}{\hbar} , \quad (2.5.3.21)$$

the integration of the expression (2.5.3.17) will be given by:

$$S_o(t) = \frac{1}{\hbar} \int_o^t dt' \{ \frac{m}{2} \dot{q}^2(t') + \frac{m \nu}{2} [\frac{\dot{a}(t')}{a(t')} - \frac{\nu}{2}] - V[q(t'), t'] - \frac{\hbar^2}{4 m a^2(t')} \} +$$

$$+ \frac{m v_o x_o}{\hbar} . \quad (2.5.3.22)$$

Taking into account the eq. (2.5.3.22) in the eq. (2.5.3.11a) and considering the eq. (2.5.3.11b), results:

$$S(x, t) = \frac{1}{\hbar} \int_o^t dt' \{ \frac{m}{2} \dot{q}^2(t') + \frac{m \nu}{2} [\frac{\dot{a}(t')}{a(t')} - \frac{\nu}{2}] - V[q(t'), t'] - \frac{\hbar^2}{4 m a^2(t')} \} + \frac{m v_o x_o}{\hbar} +$$

$$+ \frac{m \dot{q}(t)}{\hbar} \times [x - q(t)] + \frac{m}{2 \hbar} [\frac{\dot{a}(t)}{a(t)} - \frac{\nu}{2}] \times [x - q(t)]^2 . \quad (2.5.3.23)$$

The eq. (2.5.3.23) permit us, finally, to obtain the wave packet for the *SCH−E*. Indeed, considering the eqs. (2.1.1.1), (2.1.3.1b) and (2.5.3.23), we get: [15.6]

$$\Psi(x, t) = [2 \pi a^2(t)]^{-1/4} \times exp (\{ \frac{i m}{2 \hbar} [\frac{\dot{a}(t)}{a(t)} - \frac{\nu}{2}] - \frac{1}{4 a^2(t)} \} \times [x - q(t)]^2) \times$$

$$\times exp \{ \frac{i m \dot{q}(t)}{\hbar} \times [x - q(t)] + \frac{i m v_o x_o}{\hbar} \} \times$$

$$\times exp (\frac{i}{\hbar} \int_o^t dt' \{ \frac{m}{2} m \dot{q}^2(t') + \frac{m \nu}{2} [\frac{\dot{a}(t)}{a(t')} - \frac{\nu}{2}] - V[q(t'), t'] - \frac{\hbar^2}{4 m a^2(t')} \}) . \quad (2.5.3.24)$$

2.5.4. Calculation of the Feynman Propagator of the Linearized Schuch-Chung-Hartmann Equation along a Classical Trajetory

For to calculated the Feynman-de Broglie-Bohm propagator of the linearized *SCH−E* along a classical trajectory, will be following the same operational protocol of the item (2.4.4). Thus, by using the eqs. (2.1.4.1-3,5,9), we can write that: [15.6]

$$\Psi(x, t) = \int_{-\infty}^{+\infty} K(x, x_o; t, 0)\, \Psi(x_o, 0)\, dx_o\,, \quad (2.5.4.1)$$

$$\Phi(v_o, x, t) = (2\,\pi\, a_o^2)^{1/4}\, \Psi(v_o, x, t)\,, \quad (2.5.4.2)$$

$$\int_{-\infty}^{+\infty} dv_o\, \Phi^*(v_o, x', t)\, \Phi(v_o, x', t) = \left(\frac{2\,\pi\,\hbar}{m}\right) \delta(x - x')\,, \quad (2.5.4.3)$$

$$\varrho(v_o, x, t) = (2\,\pi\, a_o^2)^{1/4}\, \Phi^*(v_o, x, t)\, \Psi(v_o, x, t)\,, \quad (2.5.4.4)$$

$$\Psi(x, t) = \int_{-\infty}^{+\infty} \left\{ \left(\frac{m}{2\,\pi\,\hbar}\right) \int_{-\infty}^{+\infty} dv_o\, \Phi(v_o, x, t)\, \Phi^*(v_o, x_o, 0) \right\} \times$$

$$\times\, \Psi(x_o, 0)\, dx_o\,. \quad (2.5.4.5)$$

Comparing the eqs. (2.5.4.1,5), we have:

$$K(x, x_o, t) = \left(\frac{m}{2\,\pi\,\hbar}\right) \int_{-\infty}^{+\infty} dv_0\, \Phi(v_o, x, t)\, \Phi^*(v_o, x_o, 0)\,. \quad (2.5.4.6)$$

Substituting the eqs.(2.5.3.24) and (2.5.4.2) in the eq. (2.5.4.6), we obtain the Feynman Propagator of the linearized *SCH−E* along a classical trajetory, that we were looking for, that is [remembering that $\Phi^*(v_o, x_o, 0) = exp\left(-\frac{i\, m\, v_o\, x_o}{\hbar}\right)$]:

$$K(x, x_o; t) = \left(\frac{m}{2\,\pi\,\hbar}\right) \int_{-\infty}^{+\infty} dv_o\, \sqrt{\frac{a_0}{a(t)}} \times$$

$$\times\, exp\left(\left\{ \frac{i}{2}\, \frac{m}{\hbar} \left[\frac{\dot{a}(t)}{a(t)} - \frac{\nu}{2} \right] - \frac{1}{4\, a^2(t)} \right\} \times [x - q(t)]^2 \right) \times$$

$$\times\, exp \left\{ \frac{i\, m\, \dot{q}(t)}{\hbar} \times [x - q(t)] \right\} \times$$

$$\times\, exp\left(\frac{i}{\hbar} \int_o^t dt' \left\{ \frac{m}{2} \dot{q}^2(t') + \frac{m\,\nu}{2} \left[\frac{\dot{a}(t')}{a(t')} - \frac{\nu}{2} - V[q(t'), t'] - \frac{\hbar^2}{4\, m\, a^2(t')} \right\} \right]\,, \quad (2.5.4.7)$$

where $q(t)$ and $a(t)$ are solutions of the differential equations given by the eqs.(2.5.3.18,19).

Finally, it is important to note that putting $\nu = 0$ and $V[q(t'), t'] = 0$ into eqs. (2.5.3.18,19) and (2.5.4.7), we obtain the free Feynman propagator. [2]

2.6. The Süssmann-Hasse-Albrecht-Kostin-Nassar Equation

In 1973, D. Süssmann [20] and, in 1975, R. W. Hasse, [21] K. Albrecht, [22] and M. D. Kostin [23] proposed a non-linear Schrödinger equation, that was generalized by A. B. Nassar, in 1986, [18] to represent time dependent physical systems, given by:

$$i \hbar \frac{\partial \Psi(x, t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x, t)}{\partial x^2} +$$

$$+ \{ V(x, t) + \nu ([x - q(t)] \times [c \hat{p} + (1 - c) \langle \hat{p} \rangle] - \frac{i \hbar c}{2}) \} \times \Psi(x, t) , \quad (2.6.1)$$

where $\hat{p}$ is the operator of linear momentum:

$$\hat{p} = -i \hbar \frac{\partial}{\partial x} , \quad (2.6.2)$$

and c is a constant, where: $c = 1$, for Süssmann; $c = 1/2$, for Hasse; and $c = 0$, for Albrecht and Kostin. Besides $\Psi(x, t)$ and $V(x, t)$ are, respectively, the wavefunction and the time dependent potential of the physical system in study, $q(t) = \langle x \rangle$, and ν is a constant.

2.6.1. The Wave Function of the Süssmann-Hasse-Albrecht-Kostin-Nassar Equation

Putting the eqs. (2.1.1.1), (2.1.1.2a,b) and (2.6.2) into the eq. (2.6.1), we have: [4,15.7]

$$i \hbar (\frac{\partial \varphi}{\partial t} + i \varphi \frac{\partial S}{\partial t}) = -\frac{\hbar^2}{2m} [\frac{\partial^2 \varphi}{\partial x^2} + 2i \frac{\partial S}{\partial x} \frac{\partial \varphi}{\partial x} + i \varphi \frac{\partial^2 S}{\partial x^2} - \varphi (\frac{\partial S}{\partial x})^2] +$$

$$+ \{ V(x, t) + \nu ([x - q(t)] \times [c \hbar (\frac{\partial S}{\partial x} - \frac{i}{\varphi} \frac{\partial \varphi}{\partial x}) +$$

$$+ (1 - c) \langle \hat{p} \rangle] - \frac{1}{2} i \hbar c) \} \times \varphi(x, t) , \quad (2.6.1.1)$$

Separating the real and imaginary parts of the relation (2.6.1.1), results (remember that $\langle \hat{p} \rangle = m \langle \hat{v}_{qu} \rangle = m \langle v_{qu} \rangle =$ real):

a) <u>imaginary part</u>

$$\frac{\hbar}{\varphi} \frac{\partial \varphi}{\partial t} = -\frac{\hbar^2}{2m} (\frac{2}{\varphi} \frac{\partial S}{\partial x} \frac{\partial \varphi}{\partial x} + \frac{\partial^2 S}{\partial x^2}) - \nu [x - q(t)] \times c \frac{\hbar}{\varphi} \frac{\partial \varphi}{\partial x} - \frac{\nu}{2} \hbar c , \quad (2.6.1.2)$$

b) <u>real part</u>

$$-\hbar \frac{\partial S}{\partial t} = -\frac{\hbar^2}{2m} [\frac{1}{\varphi} \frac{\partial^2 \varphi}{\partial x^2} - \varphi (\frac{\partial \varphi}{\partial x})^2] +$$

$$+ \nu [x - q(t)] \times c \hbar \frac{\partial S}{\partial x} + V(x, t) + \nu [x - q(t)] \times (1 - c) m \langle v_{qu} \rangle . \quad (2.6.1.3)$$

2.6.2. Dynamics of the Süssmann-Hasse-Albrecht-Kostin-Nassar Equation

Now, let us see the correlation between the expressions (2.6.1.2-3) and the traditional equations of the Ideal Fluid Dynamics: [8] a) *Continuity Equation*, b) *Euler's equation* (for conservative systems) or b') *Navier-Stokes equation* (for non-conservative systems). Thus, putting the eqs. (2.1.2.1a-f) into the eq. (2.6.1.2), and using the same operational protocol of the item 2.5.2, we obtain: [4,15.7]

$$\frac{\partial \varrho}{\partial t} + \frac{\partial (\varrho v_{qu})}{\partial x} = -\nu c \{ \varrho + [x - q(t)] \frac{\partial \varrho}{\partial x} \}, \quad (2.6.2.1)$$

expression that indicates decoherence of the considered physical system represented by the Süssmann-Hasse-Albrecht-Kostin-Nassar Equation (*SHAKN−E*) [eq. (2.6.1)]; then the *Continuity Equation* it is not preserved. To preserved them, let us use the definition: [18]

$$v_{qum} = v_{qu} + \nu\, c\, [x - q(t)], \quad (2.6.2.2)$$

the *quantum velocity non-conservative*.

Multiplying the eq. (2.6.2.2) by $\varrho(x, t)$ [eq. (2.1.3.1a)], differenting the result in the variable x and remembering that $\partial q(t)/\partial x = 0$, we have: [15.7]

$$-\nu\, c\, \{ \varrho\, [x - qt(t)]\, \frac{\partial \varrho}{\partial x} \} = \frac{\partial(\varrho\, v_{qu})}{\partial x} - \frac{\partial(\varrho\, v_{qum})}{\partial x}. \quad (2.6.2.3)$$

Putting the eq. (2.6.2.3) into eq. (2.6.2.1), we obtain:

$$\frac{\partial \varrho}{\partial t} + \frac{\partial(\varrho\, v_{qum})}{\partial x} = 0, \quad (2.6.2.4)$$

expression that indicates coherence of the *SHAKN−E*.

Now, let us obtained another dynamic equation of the *SHAKN−E*. Considering the eq. (2.5.2.3):

$$<f(x, t)> = \int_{-\infty}^{+\infty} \varrho(x, t)\, f(x, t)\, dx = g(t), \quad (2.6.2.5)$$

where $\varrho(x, t)$ is given by the eq. (2.1.3.1a), results in (remembering that $\int_{-\infty}^{+\infty} exp(-z^2)\, dz = \sqrt{\pi}$, $<x> = q(t)$ and $\dot{q}(t) = d<x>/dt$):

$$<q(t)> = \int_{-\infty}^{+\infty} \varrho(x, t)\, q(t)\, dx = g(t), \quad (2.6.2.6)$$

$$\frac{\partial <v_{qu}>}{\partial x} = \frac{\partial}{\partial x} \int_{-\infty}^{+\infty} \varrho(x, t)\, v_{qu}(x, t)\, dx = \frac{\partial g(t)}{\partial x} = 0, \quad (2.6.2.7)$$

$$< v_{qum} > = < v_{qu} >. \quad (2.6.2.8)$$

Now, differentiating the eq. (2.6.1.3) with respect x, and using the eqs. (2.1.2.1c-f) and (2.6.2.2,6-8), we have: [15.7]

$$\frac{\partial v_{qum}}{\partial t} + v_{qum} \frac{\partial v_{qum}}{\partial x} =$$

$$-\nu\, [c\, (v_{qu} - v_{qum}) + <v_{qum}>] - \frac{1}{m} \frac{\partial}{\partial x} (V + V_{qu}). \quad (2.6.2.9)$$

We observe that the eq. (2.6.2.9) has the aspect of the *Navier-Stokes Equation* [8] for a real fluid in movement.

Considering the *substantive differentiation* (local plus convective) or *hydrodynamic differentiation*, given by the eqs. (2.1.2.5a,b) and inserting into eq. (2.6.2.9), results:

$$\frac{d^2x}{dt^2} + \nu\, [c\, (v_{qu} - v_{qum}) + <v_{qum}>] = \frac{1}{m} \frac{\partial}{\partial x} [\, V(x, t) + V_{qu}(x, t)\,], \quad (2.6.2.10)$$

what has a form of the *Dissipative Second Newton Law*, being the terms of the second member, respectively, the *classical newtonian force* and the *quantum bohmnian force*.

2.6.3 The Quantum Wave Packet of the Linearized Süssmann-Hasse-Albrecht-Kostin-Nassar Equation along a Classical Trajetory

In order to find the quantum wave packet of the linearized Süssmann-Hasse-Albrecht-Kostin-Nassar Equation (*SHAKN−E*) along a trajectory classic, we must integrated the eq. (2.6.2.4). So, using the same protocol of the item 2.5.3, we have: [15.7]

$$v_{qum}(x, t) = \frac{\dot{a}(t)}{a(t)}\,[x - q(t)] + \dot{q}(t)\,. \qquad (2.6.3.1)$$

Considering the eqs. (2.6.2.2) and (2.6.3.1), results:

$$v_{qu}(x, t) \equiv \frac{dx(t)}{dt} = \left[\frac{\dot{a}(t)}{a(t)} - \nu\, c\right] \times [x - q(t)] + \dot{q}(t)\,. \qquad (2.6.3.2)$$

We observe that the integration of the eq. (2.6.3.2) given the *bohmnian quantum trajectory* of the physical system represented by *SHAKN−E*.

To obtain the quantum wave packet [$\Psi(x\,,\,t)$] of the *SHAKN−E* given by eq. (2.6.1), let us expand the functions S(x, t), V(xt), and V_{qu}(x, t) around of $q(t) = \langle x \rangle$ up to *second Taylor order*. [9] In this way, using the eq. (2.1.3.14) we have:

$$S(x, t) = S[q(t), t] + S'[q(t), t] \times [x - q(t)] + \frac{1}{2}\, S''[q(t)\ t] \times [x - q(t)]^2, \qquad (2.6.3.3)$$

$$V(x, t) = V[q(t), t] + V'[q(t), t] \times [x - q(t)] +$$

$$+\ \frac{1}{2}\, V''[q(t), t] \times [x - q(t)]^2, \qquad (2.6.3.4)$$

$$V_{qu}(x, t) = V_{qu}[q(t), t] + V'_{qu}[q(t), t] \times [x - q(t)] +$$

$$+\ \frac{1}{2}\, V''_{qu}[q(t), t] \times [x - q(t)]^2, \qquad (2.6.3.5a)$$

$$V_{qu}(x, t) = \frac{\hbar^2}{4\, m\, a^2(t)} - \frac{\hbar^2}{8\, m\, a^4(t)} \times [x - q(t)]^2\,. \qquad (2.6.3.5b)$$

Differenting the eq. (2.6.3.3) in the variable x, multiplying the result by $\hbar/m$ and using the eqs. (2.1.2.1c,d) and (2.6.3.2), results:

$$\frac{\hbar}{m}\,\frac{\partial S(x, t)}{\partial x} = \frac{\hbar}{m}\,\{\, S'[q(t), t] + S''[q(t), t] \times [x - q(t)]\,\} = v_{qu}(x, t) =$$

$$= \left[\frac{\dot{a}(t)}{a(t)} - \nu\, c\right] \times [x - q(t)] + \dot{q}(t) \ \ \rightarrow$$

$$S'[q(t), t] = \frac{m\,\dot{q}(t)}{\hbar}\,, \quad S''[q(t), t] = \frac{m}{\hbar}\left[\frac{\dot{a}(t)}{a(t)} - \nu\, c\right]. \qquad (2.6.3.6a,b)$$

Substituting the eqs. (2.6.3.6a,b) into eq. (2.6.3.3), we have:

$$S(x, t) = S_0(t) + \frac{m\,\dot{q}(t)}{\hbar} \times [x - q(t)] + \frac{m}{2\,\hbar}\left[\frac{\dot{a}(t)}{a(t)} - \nu\, c\right] \times [x - q(t)]^2\,, \qquad (2.6.3.7a)$$

where [see eq. (2.1.3.11b)]:

$$S_o(t) \equiv S[q(t), t] \,. \quad (2.6.3.7b)$$

Differenting the eq. (2.6.3.7a) in relation to the time t and using the eq. (2.1.3.2), results (remember that $\partial x/\partial t = 0$): [15.7]

$$\frac{\partial S(x, t)}{\partial t} = \dot{S}_o(t) - \frac{m\,\dot{q}^2(t)}{\hbar} + \frac{m}{\hbar}\{\ddot{q}(t) - \dot{q}(t)\,[\frac{\dot{a}(t)}{a(t)} - \nu\, c\,]\} \times [x - q(t)] +$$

$$+ \frac{m}{2\,\hbar}[\frac{\ddot{a}(t)}{a(t)} - \frac{\dot{a}^2(t)}{a^2(t)}] \times [x - q(t)]^2 \,. \quad (2.6.3.8)$$

Using the eqs. (2.1.2.1c-e) and (2.6.1.3), we obtain:

$$-\,\hbar\,\frac{\partial S}{\partial t} = -\,\frac{\hbar^2}{2\,m}[\,\frac{1}{\varphi}\,\frac{\partial^2\varphi}{\partial x^2} - \varphi\,(\frac{\partial S}{\partial x})^2\,] + V(x\,,t) + \nu\,[x - q(t)] \times c\,\hbar\,\frac{\partial S}{\partial x} +$$

$$+\,\nu\,[x - q(t)]\,(1 - c)\,m < v_{qu} > \;\rightarrow$$

$$\hbar\,\frac{\partial S}{\partial t} + \frac{1}{2}\,m\,v_{qu}^2 + V(x, t) + V_{qu}(x, t) +$$

$$+\,\nu\,m\,[x - q(t)] \times [c\,v_{qu} + (1 - c) < v_{qu} >] = 0 \,. \quad (2.6.3.9)$$

Inserting the eqs. (2.6.3.2,4,6a,b) into eq. (2.6.3.9), ordering the result in potencies of $[x - q(t)]$, and considering that $[x - q(t)]^0 = 1$, we have: [15.7]

$$\{\hbar\,\dot{S}_o(t) - \frac{m}{2}\dot{q}^2(t) + \nu\,m\,(1 - c)\,\dot{q}(t) + V[q(t), t] + \frac{\hbar^2}{4\,m\,a^2(t)}\} \times [x - q(t)]^0 +$$

$$+\,\{m\,\ddot{q}(t) + m\,\nu\,c\,\dot{q}(t) + V'[q(t), t]\} \times [x - q(t)] +$$

$$+\,\{\frac{m}{2}\,\frac{\ddot{a}(t)}{a(t)} - \frac{m\,\nu^2\,c^2}{2} + \frac{1}{2}\,V''[q(t), t] - \frac{\hbar^2}{8\,m\,a^4(t)}\} \times [x - (q)t)]^2 = 0 \,. \quad (2.6.3.10)$$

As the above relation [eq. (2.6.3.10)] is an identically null polynomium, the coefficients of the potencies must be all equal to zero, that is:

$$\dot{S}_o(t) = \frac{1}{\hbar}\{\,\frac{m}{2}\dot{q}^2(t) - V[q(t), t] - \nu\,m\,(1 - c)\,\dot{q}(t) - \frac{\hbar^2}{4\,m\,a^2(t)}\}\,, \quad (2.6.3.11)$$

$$\ddot{q}(t) + \nu\,c\,\dot{q}(t) + \frac{1}{m}\,V'[q(t), t] = 0\,, \quad (2.6.3.12)$$

$$\ddot{a}(t) + \{\,\frac{1}{m}\,V''[q(t), t] - \nu\,c^2\,\} \times a(t) = \frac{\hbar^2}{4\,m^2\,a^3(t)}\,. \quad (2.6.3.13)$$

Assuming that the following initial conditions are obeyed [see eqs. (2.1.3.21a-d):

$$q(0) = x_o\,,\quad \dot{q}(0) = v_o\,,\quad a(0) = a_o\,,\quad \dot{a}(0) = b_o\,, \quad (2.6.3.14a\text{-}d)$$

and that [see eqs. (2.1.2.1c,d) and (2.6.3.7b)]:

$$S_o(0) = \frac{m\, v_o\, x_o}{\hbar}, \quad (2.6.3.15)$$

the integration of the expression (2.6.3.11) will be given by:

$$S_o(t) = \frac{1}{\hbar} \int_o^t dt' \, \{ \frac{m}{2} \dot{q}^2(t') - \nu\, m\, (1 - c)\, \dot{q}(t') - V[q(t'), t'] - \frac{\hbar^2}{4\, m\, a^2(t')} \} +$$

$$+ \frac{m\, v_o\, x_o}{\hbar}. \quad (2.6.3.16)$$

Taking into account the eq. (2.6.3.16) in the eq. (2.6.3.7a) and considering the eq. (2.6.3.7b), results:

$$S(x, t) = \frac{1}{\hbar} \int_o^t dt' \, \{ \frac{m}{2} \dot{q}^2(t') - \nu\, m\, (1 - c)\, \dot{q}(t') - V[q(t'), t'] - \frac{\hbar^2}{4\, m\, a^2(t')} \} + \frac{m\, v_o\, x_o}{\hbar} +$$

$$+ \frac{m\, \dot{q}(t)}{\hbar} \times [x - q(t)] + \frac{m}{2\, \hbar} [\frac{\dot{a}(t)}{a(t)} - \nu\, c] \times [x - q(t)]^2 . \quad (2.6.3.17)$$

The eq. (2.6.3.17) permit us, finally, to obtain the wave packet for the *SHAKN−E*. Indeed, considering the eqs. (2.1.1.1), (2.1.3.1b) and (2.6.3.17), we get: [15.7]

$$\Psi(x, t) = [2\, \pi\, a^2(t)]^{-1/4} \times exp\, (\{ \frac{i\, m}{2\, \hbar} [\frac{\dot{a}(t)}{a(t)} - \nu\, c] - \frac{1}{4\, a^2(t)} \} \times [x - q(t)]^2) \times$$

$$\times\, exp\, \{ \frac{i\, m\, \dot{q}(t)}{\hbar} \times [x - q(t)] + \frac{i\, m\, v_o\, x_o}{\hbar} \} \times$$

$$\times\, exp\, (\frac{i}{\hbar} \int_o^t dt' \, \{ \frac{m}{2}\, m\, \dot{q}^2(t') - \nu\, m\, (1 - c)\, \dot{q}(t') -$$

$$- V[q(t'), t'] - \frac{\hbar^2}{4\, m\, a^2(t')} \}). \quad (2.6.3.18)$$

2.6.4. Calculation of the Feynman Propagator of the Linearized Süssmann-Hasse-Albrecht-Kostin-Nassar Equation along a Classical Trajetory

For to calculated the Feynman-de Broglie-Bohm propagator of the linearized *SHAKN−E* along a classical trajectory, will be following the same operational protocol of the item 2.5.4. Thus, by using the eqs. (2.1.4.1-3,5,9), we can write that: [15.7]

$$\Psi(x, t) = \int_{-\infty}^{+\infty} K(x, x_o; t, 0)\, \Psi(x_o, 0)\, dx_o\,, \quad (2.6.4.1)$$

$$\Phi(v_o, x, t) = (2\, \pi\, a_o^2)^{1/4}\, \Psi(v_o, x, t)\,, \quad (2.6.4.2)$$

$$\int_{-\infty}^{+\infty} dv_o\, \Phi^*(v_o, x', t)\, \Phi(v_o, x, t) = (\frac{2\, \pi\, \hbar}{m})\, \delta(x - x')\,, \quad (2.6.4.3)$$

$$\varrho(v_o, x, t) = (2\, \pi\, a_o^2)^{1/4}\, \Phi^*(v_o, x, t)\, \Psi(v_o, x, t)\,, \quad (2.6.4.4)$$

$$\Psi(x, t) = \int_{-\infty}^{+\infty} \{ (\frac{m}{2 \pi \hbar}) \int_{-\infty}^{+\infty} dv_o \, \Phi(v_o, x, t) \, \Phi^*(v_o, x_o, 0) \} \times$$

$$\times \Psi(x_o, 0) \, dx_o \, . \quad (2.6.4.5)$$

Comparing the eqs. (2.6.4.1,5), we have:

$$K(x, x_o, t) = (\frac{m}{2 \pi \hbar}) \int_{-\infty}^{+\infty} dv_0 \, \Phi(v_o, x, t) \, \Phi^*(v_o, x_o, 0) \, . \quad (2.6.4.6)$$

Substituting the eqs.(2.6.3.18) and (2.6.4.2) in the eq. (2.6.4.6), we obtain the Feynman Propagator of the linearized *SHAKN−E* along a classical trajetory, that we were looking for, that is [remembering that $\Phi^*(v_o, x_o, 0) = exp\,(- \frac{i\, m\, v_o\, x_o}{\hbar})$]:

$$K(x, x_o; t) = (\frac{m}{2 \pi \hbar}) \int_{-\infty}^{+\infty} dv_o \sqrt{\frac{a_0}{a(t)}} \times$$

$$\times\, exp\, (\{ \frac{i}{2} \frac{m}{\hbar} [\frac{\dot{a}(t)}{a(t)} - \frac{\nu}{2}] - \frac{1}{4\, a^2(t)} \} \times [x - q(t)]^2) \times$$

$$\times\, exp\, \{ \frac{i\, m\, \dot{q}(t)}{\hbar} \times [x - q(t)] \} \times$$

$$\times\, exp\, (\frac{i}{\hbar} \int_o^t dt' \, \{ \frac{m}{2} \dot{q}^2(t') - \nu\, m(1 - c)\, \dot{q}(t') - V[q(t'), t'] - \frac{\hbar^2}{4\, m\, a^2(t')} \}] \, , \quad (2.6.4.7)$$

where $q(t)$ and $a(t)$ are solutions of the differential equations given by the eqs.(2.6.3.12,13).

Finally, it is important to note that putting $\nu = 0$ and $V[q(t'), t'] = 0$ into eqs. (2.6.3.12,13) and (2.6.4.7), we obtain the free Feynman propagator. [2]

2.7. The Schrödinger-Nassar Equation for and Extended Electron

2.7.1. Introduction

In 1892, [24] Hendrik Antoon Lorentz and, in 1905, [25] Max Abraham argued that when an electron (with velocity v and charge e), is accelerated, there are additional forces acting due to the electron's own electromagnetic field. However, the so-called *Lorentz-Abraham Equation* for a point-charge electron:

$$m\, dv/dt = (2\, e^2)/(3\, c^2)\, d^2v/dt^2 + F_{ext}, \quad (2.7.1.1)$$

it was found to be unsatisfactory because, for $F_{ext} = 0$, it admits runaway solutions. These solutions clearly violate the law of inertia.

Since the seminal works of Lorentz and Abraham, many papers and textbooks have given great consideration to the proper equation of motion of an electron. [26]-[34] The problematic runaway solutions were circumverted by Sommerfeld [28] and Page [29] adopting an *electron extended model* (*EEM*). In the nonrelativistic case of a sphere with uniform surface charge, such an electron obeys in good approximation the difference-differential equantion: [30]-[32]

$$m\, dv/dt = (e^2)/(3\, L^2\, c)\, [v(t - 2\, L/c) - v(t)] + F_{ext}. \quad (2.7.1.2)$$

This *EEM* model is finite and causal if the diameter L of the electron is larger than the classical electron radius $r_e = e^2/(m\, c^2)$. We will analyze here only the case of the sphere with uniform surface charge; the case of a volume charge is considerably more complicated and adds nothing to the understanding of the problem.

The dynamics of charges is a key example of the importance of obeying the validity limits of a physical theory. If classical equations can no longer be trusted at distances of the order (or below) the Compton wavelength, what is the Schrödinger equation that can replace equation (2.7.1.2)? Within *QED*, workers have not been able to derive an equation of motion and it is unclear whether *QED* can actually produce an equation of motion at all.

By using the Quantum Mechanical of the de Broglie-Bohm, [4] Nassar [35] propose an answer to this problem in the nonrelativistic regime. So, the Schrödinger-Nassar Equation for an Extended Electron (*SN−EEE*), is given by:

$$\mathrm{i}\,\hbar\,\frac{\partial \Psi(x,t)}{\partial t} = -\,\frac{\hbar^2}{2\,m}\,\frac{\partial^2\,\Psi(x,t)}{\partial x^2} +$$

$$+\,(\,i\,\hbar\,\alpha\,\ell n\,[\,\frac{\psi(x,t-2\,L/c)\,\psi^*(x,t)}{\psi^*(x,t-2\,L/c)\,\psi(x,t)}\,]\,)\times\Psi(x,t)\,, \quad (2.7.1.3a)$$

where:

$$\alpha = \frac{e^2}{6\,m\,L^2\,c}, \quad (2.7.1.3b)$$

is a constant, beeing $\Psi(x, t)$ and $\Psi(x, t-2\,L/c)$ wave functions.

2.7.2. Quantum Wave Function of the Schrödinger-Nassar Equation for and Extended Electron

Now, let us to obtain the quantum wave packet for the Schrödinger-Nassar Equation for an Extended Electron (*SN−EEE*). Initially, let us to write the wavefunction $\psi(x, t)$ and $\psi(x, t-2\,L/c)$ in the polar form, defined by the Madelung-Bohm transformation [see eq. (2.1.1.1)], we get:

$$\Psi(x,t) = \varphi(x,t)\,exp\,[i\,S(x,t)]\,, \quad (2.7.2.1a)$$

$$\Psi(x,t-2\,L/c) = \varphi(x,t)\,exp\,[i\,S(x,t-2\,L/c)]\,, \quad (2.7.2.1b)$$

where $S(x\,,t)$ is the *classical action* and $\varphi(x, t)$ will be defined in what follows.

Calculating the derivatives, temporal and spatial, of (2.7.2.1a), we get: [15.4]

$$i\,\hbar\,(\,\frac{\partial\varphi}{\partial t} + i\,\varphi\,\frac{\partial S}{\partial t}\,) = -\,\frac{\hbar^2}{2\,m}\,[\,\frac{\partial^2\varphi}{\partial x^2} + 2\,i\,\frac{\partial S}{\partial x}\,\frac{\partial\varphi}{\partial x} + i\,\varphi\,\frac{\partial^2 S}{\partial x^2} - \varphi\,(\frac{\partial S}{\partial x})^2\,] -$$

$$-\,(\,2\,\hbar\,\alpha\,[\,S(x,t-2\,L/c) - S(x,t)\,]\,)\times\varphi(x,t)\,. \quad (2.7.2.2)$$

Taking the real and imaginary parts of (2.7.2.2), we obtain:

a) <u>imaginary part</u>

$$\frac{\partial\varphi(x,t)}{\partial t} = -\,\frac{\hbar}{2\,m}\,[\,2\,\frac{\partial S(x,t)}{\partial x}\,\frac{\partial\varphi(x,t)}{\partial x} + \varphi(x,t)\,\frac{\partial^2 S(x,t)}{\partial x^2}\,]\,, \quad (2.7.2.3)$$

b) <u>real part</u>

$$-\,\hbar\,\frac{\partial S(x,t)}{\partial t} = -\,\frac{\hbar^2}{2\,m}\,\{\,\frac{1}{\varphi}\,\frac{\partial^2\varphi(x,t)}{\partial x^2} - [\,\frac{\partial S(x,t)}{\partial x}\,[^2\,\} +$$

$$+ \{ \frac{2 \hbar \alpha}{m} [S(x, t) - S(x, t - 2L/c)] \} . \qquad (2.7.2.4)$$

2.7.3. Dynamics of the Schrödinger-Nassar Equation for an Extended Electron

Now, let us see the correlation between (2.7.2.3-4) and the traditional equations of the Fluid Dynamics [8]: a) *Continuity Equation*, b) *Euler Equation* (for conservative systems) or b') *Navier-Stokes Equation* (for non-conservative systems). To do this let us perform the following correspondences [see eqs. (2.1.2.a-f)]:

Quantum density probability: $| \Psi(x, t) |^2 = \Psi^*(x, t)\, \Psi(x, t) \quad \longleftrightarrow$

Quantum mass density: $\varrho(x, t) = \varphi^2(x, t) \quad \longleftrightarrow \quad \sqrt{\varrho} = \varphi \,, \qquad (2.7.3.1a,b)$

Gradient of the wave function: $\frac{\hbar}{m} \frac{\partial S(x, t)}{\partial x} \quad \longleftrightarrow$

Quantum velocity: $v_{qu}(x, t) \equiv v_{qu} \,, \qquad (2.7.3.2a,b)$

Gradient of the wave function extended: $\frac{\hbar}{m} \frac{\partial S(x, t - 2\,L/c)}{\partial x} \quad \longleftrightarrow$

Quantum velocity extended: $v_{qu}(x, t - 2\,L/c) \equiv v_{que} \,. \qquad (2.7.3.3a,b)$

Bohm quantum potential:

$$V_{qu}(x, t) \equiv V_{qu} = - \left(\frac{\hbar^2}{2\,m\,\varphi}\right) \frac{\partial^2 \varphi}{\partial x^2} = - \frac{\hbar^2}{2\,m} \frac{1}{\sqrt{\varrho}} \frac{\partial^2 \sqrt{\varrho}}{\partial x^2} \,, \qquad (2.7.3.4a,b)$$

Using the same operational protocol of the item (2.1.2), results: [4]

$$\frac{\partial \varrho}{\partial t} + \frac{\partial (\varrho\, v_{qu})}{\partial x} = 0 \,, \qquad (2.7.3.5)$$

expression that indicates <u>coherence</u> of the considered physical system represented by the *SN−EEE* [see eq. (2.7.1.3a)]; then the *Continuity Equation* it is preserved.

Now, let us obtained another dynamic equation of the *SN−EEE*. So, differentiating the eq. (2.7.2.4) with respect x and using the eqs. (2.1.2.1c-f), we obtain: [15.4]

$$\hbar \frac{\partial S(x, t)}{\partial t} + \frac{1}{2} m\, v_{qu}^2 + V_{ee} + V_{qu} = 0, \qquad (2.7.3.6a)$$

where:

$$V_{ee} = 2\, \hbar\, \alpha\, [S(x, t) - S(x, t - 2L/c)] \,, \qquad (2.7.3.6b)$$

is the *potential of the extended electron*.

Differentiating the eq. (2.7.3.6a) with respect x and using the eqs. (2.1.2.1e,f) and the eq. (2.7.3.6b), we have: [15.4]

$$\frac{\partial v_{qu}}{\partial t} + v_{qu} \frac{\partial v_{qu}}{\partial x} = - \frac{2\,\alpha}{m} [v_{qu}(t - 2\,L/c) - v_{qu}(t)] - \frac{1}{m} \frac{\partial}{\partial x} V_{qu}. \qquad (2.7.3.7)$$

We observe that the eq. (2.7.3.7) has the aspect of the *Navier-Stokes Equation* [8] for a fluid in movement.

Now, considering the "substantive differentiation" (local plus convective) or "hidrodynamical differention", given by the eqs. (2.1.2.5a,b) and inserting in the eq. (2.7.3.7), results:

$$m \frac{d^2x(t)}{dt^2} \equiv m \frac{d v_{qu}}{dt} = (e^2)/(3 L^2 c) [v_{qu}(t - 2 L/c) - v_{qu}(t)] + F_{qu} , \quad (2.7.3.8a)$$

that has the same form of the *Dissipative Second Newton Law*, where:

$$F_{qu} = - \frac{\partial}{\partial x} V_{qu} . \quad (2.7.3.8b)$$

is the *quantum bohmnian force*. We note that (2.7.3.8a) is the quantum form of the *Sommerfeld-Page's Equation* [see eq. (2.7.1.2)] in the absence of external forces ($F_{ext} = 0$).

2.7.4. Calculation of the Quantum Wave Packet of the Schrödinger-Nassar Equation for an Extended Electron along a Classical Trajectory

In order to find the quantum wave packet of the Schrödinger-Nassar Equation for an Extended Electron (*SN−EEE*) along a classical trajectory, we must calculate the v_{qu}. So, using the same operational protocol of the item (2.1.3), we have: [4,15.4]

$$v_{qu} (x, t) = \frac{\dot{a}(t)}{a(t)} [x - q(t)] + \dot{q}(t). \quad (2.7.4.1)$$

We observe that te integration of the eq. (2.7.4.1) given the *bohmnian quantum trajectory* of the *SN−EEE*.

To obtain the quantum wave packet [$\Psi(x, t)$] of the *SN−EEE* given by (2.7.1.3a,b), let us expand the functions $S(x, t)$, $S(x, t - 2 L/c)$ and $V_{qu}(x, t)$ around of $q(t)$ up to second Taylor order. In this way, using the eq. (2.1.3.14) we have:

$$S(x, t) = S[q(t), t] + S'[q(t), t] \times [x - q(t)] + \frac{1}{2} S''[q(t), t] \times [x - q(t)]^2 , \quad (2.7.4.2a)$$

$$S(x, t - 2 L/c) = S[q(t), t - 2 L/c] + S'[q(t), t - 2 L/c] \times [x - q(t)] +$$

$$+ \frac{1}{2} S''[q(t), t - 2 L/c] \times [x - q(t)]^2 , \quad (2.7.4.2b)$$

$$V_{qu}(x, t) = V_{qu}[q(t), t] + V_{qu}'[q(t), t] \times [x - q(t)] + \frac{V_{qu}''[q(t), t]}{2} \times [x - q(t)]^2 . \quad (2.7.4.3a)$$

$$V_{qu}(x, t) = \frac{\hbar^2}{4 m a^2(t)} - \frac{\hbar^2}{8 m a^4(t)} \times [x - q(t)]^2 . \quad (2.7.4.3a)$$

Differentiating the eqs. (2.7.4.2a,b) in the variable x, multiplying the result by $\frac{\hbar}{m}$, using the eqs. (2.1.2.1c,d) and (2.7.4.1), results: [15.4]

$$\frac{\hbar}{m} \frac{\partial S(x, t)}{\partial x} = \frac{\hbar}{m} \{ S'[q(t), t] + S''[q(t), t] \times [x - q(t)] \} =$$

$$= v_{qu}(x, t) = [\frac{\dot{a}(t)}{a(t)}] \times [x - q(t)] + \dot{q}(t) \quad \rightarrow$$

$$S'[q(t), t] = \frac{m}{\hbar}\dot{q}(t)\ , \quad S''[q(t), t] = \frac{m}{\hbar}\,\frac{\dot{a}(t)}{a(t)}\ , \qquad (2.7.4.4a,b)$$

$$\frac{\hbar}{m}\,\frac{\partial S(x, t - 2\,L/c)}{\partial x} = \frac{\hbar}{m}\,(\,S'[q(t), t - 2\,L/c] + S''[q(t), t - 2\,L/c] \times [x - q(t)]\,) =$$

$$= v_{que}(x, t - 2\,L/c) = \frac{\dot{a}(t - 2\,L/c)}{a(t)} \times [x - q(t)] + \dot{q}(t - 2\,L/c) \quad \rightarrow$$

$$S'[q(t), t - 2\,L/c] = \frac{m\,\dot{q}(t - 2\,L/c)}{\hbar}\ , \quad S''[q(t), t - 2\,L/c] = \frac{m}{\hbar}\,\frac{\dot{a}(t - 2\,L/c)}{a(t)}. \qquad (2.7.4.4c,d)$$

Substituting the eqs. (2.7.4.4a-d) in the eqs. (2.7.4.2a,b), we obtain:

$$S(x, t) = S_o(t) + \frac{m\,\dot{q}(t)}{\hbar} \times [x - q(t)] + \frac{m}{2\,\hbar}\,\frac{\dot{a}(t)}{a(t)} \times [x - q(t)]^2\ , \qquad (2.7.4.5a)$$

$$S(x, t - 2\,L/c) = S_o(t - 2\,L/c) + \frac{m\,\dot{q}(t - 2\,L/c)}{\hbar} \times [x - q(t)] +$$

$$+ \frac{m}{2\,\hbar}\,\frac{\dot{a}(t - 2\,L/c)}{a(t)} \times [x - q(t)]^2\ , \qquad (2.7.4.5b)$$

where [see eq. (2.1.3.11b)]:

$$S_o(t) \equiv S[q(t), t]\ , \qquad (2.7.4.5c)$$

$$S_o(t - 2\,L/c) \equiv S[q(t), t - 2\,L/c]\ , \qquad (2.7.4.5d)$$

are the *classical actions*.

Differentiating the (2.7.4.5a) with respect to t, we have (remembering that $\frac{\partial x}{\partial t} = 0$): [15.4]

$$\frac{\partial S}{\partial t} = \dot{S}_o(t) - \frac{m\,\dot{q}^2(t)}{\hbar} + \frac{m}{\hbar}\,\{\,\ddot{q}(t) - \dot{q}(t)\,[\,\frac{\dot{a}(t)}{a(t)}\,]\,\} \times [x - q(t)] +$$

$$+ \frac{m}{2\,\hbar}\,[\,\frac{\ddot{a}(t)}{a(t)} - \frac{\dot{a}^2(t)}{a^2(t)}\,] \times [x - q(t)]^2\ . \qquad (2.7.4.6)$$

Using the eqs. (2.1.2.1c-e) and (2.7.2.4), results:

$$\hbar\,\frac{\partial S(x, t)}{\partial t} + frac12\ m\,v_{qu}^2 +$$

$$+ 2\,\hbar\,\alpha\,[S(x, t) - S(x, t - 2L/c)] + V_{qu} = 0\ . \qquad (2.7.4.7)$$

Inserting the eqs. (2.7.4.1), (2.7.4.5a,b) and (2.7.4.6), in the eq. (2.7.4.7), ordering the result in potencies of $[x, q(t)]$, and remembering that $[x, q(t)]^0 = 1$, we have: [15.4]

$$\{\,\hbar\,\dot{S}_o(t) - \frac{m}{2}\dot{q}(t)^2 + 2\,\hbar\,\alpha\,[\,S_0(t) - S_0(t - 2\,L/c)\,] + \frac{\hbar^2}{4\,m\,a^2(t)}\,\} \times [x - q(t)]^o +$$

$$+ \{\,m\,\ddot{q}(t) + 2\,\alpha\,m\,[\,\dot{q}(t) - \dot{q}(t - 2\,L/c)\,)\,\} \times [x - q(t)] +$$

$$+ \left\{ \frac{m}{2} \frac{\ddot{a}(t)}{a(t)} + m\,\alpha \left[\frac{\dot{a}(t)}{a(t)} - \frac{\dot{a}(t - 2\,L/c)}{a(t)} \right] - \frac{\hbar^2}{8\,m\,a^4(t)} \right\} \times [x - q(t)]^2 = 0\,. \quad (2.7.4.8)$$

As the eq. (2.7.4.8) is an identically null polynomium, all coefficients of the potencies must be all equal to zero, that is:

$$\dot{S}_o(t) = \frac{1}{\hbar} \left\{ \frac{1}{2} m\,\dot{q}(t)^2 - 2\,\hbar\,\alpha\,[\,S_0(t - 2\,L/c) - S_0(t)\,] - \frac{\hbar^2}{4\,m\,a^2(t)} \right\}, \quad (2.7.4.9)$$

$$\ddot{q}(t) - 2\,\alpha\,[q(t - 2\,L/c) - \dot{q}(t)] = 0\,, \quad (2.7.4.10)$$

$$\ddot{a}(t) - 2\,\alpha\,[\dot{a}(t - 2\,L/c) - \dot{a}(t)] = \frac{\hbar^2}{4\,m^2\,a^3(t)}\,. \quad (2.7.4.11)$$

Assuming that the following initial conditions are obeyed [see eqs. (2.1.3.21a-d)]:

$$q(0) = x_o\,, \quad \dot{q}(0) = v_o\,, \quad a(0) = a_o\,, \quad \dot{a}(0) = b_o\,, \quad (2.7.4.12a\text{-}d)$$

and that [see eqs. (2.1.2.1c,d):

$$S_o(0) = \frac{m\,v_o\,x_o}{\hbar}\,, \quad (2.7.4.13)$$

the integration of (2.7.4.9) gives:

$$S_o(t) = \frac{1}{\hbar} \int_o^t dt' \left\{ \frac{1}{2} m\,\dot{q}^2(t') + 2\,\alpha\,[\,S_0(t') - S_0(t' - 2\,L/c)\,] - \right.$$

$$\left. - \frac{\hbar^2}{4\,m\,a^2(t')} \right\} + \frac{m\,v_o\,x_o}{\hbar}. \quad (2.7.4.14)$$

Taking the eqs. (2.7.4.14) in the eq. (2.7.4.5a) and considering the eq. (2.7.4.5b), results:

$$S(x,\,t) = \frac{1}{\hbar} \int_o^t dt' \left\{ \frac{1}{2} m\,\dot{q}^2(t') + 2\,\hbar\,\alpha\,[\,S_0(t') - S_0(t' - 2\,L/c)\,] - \frac{\hbar^2}{4\,m\,a^2(t')} \right\} +$$

$$+ \frac{m\,v_o\,x_o}{\hbar} + \frac{m\,\dot{q}(t)}{\hbar} \times [x - q(t)] + \frac{m}{2\,\hbar} \frac{\dot{a}(t)}{a(t)} \times [x - q(t)]^2\,. \quad (2.7.4.15)$$

The above result permit us, finally, to obtain the wave packet for the *SN−EEE* equation. Indeed, considering the eqs. (2.1.1.1), (2.1.3.1b), and (2.7.4.15), we get: [15.4]

$$\Psi(x,\,t) = [2\,\pi\,a^2(t)]^{-1/4}\,exp \left\{ \left[\frac{i\,m}{2\,\hbar} \frac{\dot{a}(t)}{a(t)} - \frac{1}{4\,a^2(t)} \right] \times [x - q(t)]^2 \right\} \times$$

$$\times\,exp \left\{ \frac{i\,m\,\dot{q}(t)}{\hbar} \times [x - q(t)] + \frac{i\,m\,v_o\,x_o}{\hbar} \right\} \times$$

$$\times\,exp \left\{ \frac{i}{\hbar} \int_o^t dt' \left(\frac{1}{2} m\,\dot{q}^2(t') + \right. \right.$$

$$+ 2 \hbar \alpha [S_0(t') - S_0(t' - 2 L/c)] - \frac{\hbar^2}{4 m a^2(t')}) \} . \quad (2.7.4.16)$$

2.7.5. **Calculation of the Feynman-de Broglie-Bohm Propagator of the Schrödinger-Nassar Equation for an Extended Electron**

For to calculate the Feynman-de Broglie-Bohm propagator of the linearized *SN−EEE* along a classical trajectory, will be following the same operational protocol of the item 2.6.4. Thus, by using the eqs. (2.1.4.1-3,5,9), we can write that: [15.4]

$$\Psi(x, t) = \int_{-\infty}^{+\infty} K(x, x_o, t)\, \psi(x_o, 0)\, dx_o , \quad (2.7.5.1)$$

$$\Phi(v_o, x, t) = (2 \pi a_o^2)^{1/4}\, \Psi(v_o, x, t) , \quad (2.7.5.2)$$

$$\int_{-\infty}^{+\infty} dv_o\, \Phi^*(v_o, x, t)\, \Phi(v_o, x', t) = (\frac{2 \pi \hbar}{m})\, \delta(x - x') , \quad (2.7.5.3)$$

$$\varrho(v_o, x, t) = (2 \pi a_o^2)^{-1/4}\, \Phi^*(v_o, x, t)\, \Psi(v_o, x, t) , \quad (2.7.5.4)$$

$$\Psi(x, t) = \int_{-\infty}^{+\infty} \{ (\frac{m}{2 \pi \hbar}) \int_{-\infty}^{+\infty} dv_o\, \Phi(v_o, x, t) \times$$

$$\times\, \Phi^*(v_o, x_o, 0) \} \times \Psi(x_o, 0)\, dx_o . \quad (2.7.5.5)$$

Comparing (2.7.5.1) and (2.7.5.5), we have:

$$K(x, x_o, t) = \frac{m}{2 \pi \hbar} \int_{-\infty}^{+\infty} dv_o\, \Phi(v_o, x, t)\, \Phi^*(v_o, x_o, 0) . \quad (2.7.5.6)$$

Substituting the eqs. (2.7.4.16) and (2.7.5.2) into the eq. (2.7.5.6), we finally obtain the Feynman-de Broglie-Bohm Propagator of the Schrödinger-Nassar Equation for an Electron Extended (*SN−EEE*), remembering that $\Phi^*(v_o, x_o, 0) = exp\,(- \frac{i\, m\, v_o\, x_o}{\hbar})$:

$$K(x, x_o; t) = \frac{m}{2 \pi \hbar} \int_{-\infty}^{+\infty} dv_o \sqrt{\frac{a_o}{a(t)}} \times$$

$$\times\, exp\, \{ [\frac{i}{2} \frac{m}{\hbar} \frac{\dot{a}(t)}{a(t)} - \frac{1}{4 a^2(t)}] \times [x - q(t)]^2 \} \times$$

$$\times\, exp\, \{ \frac{i\, m\, \dot{q}(t)}{\hbar} [x - q(t)] \} \times$$

$$\times\, exp\, [\frac{i}{\hbar} \int_o^t dt' \{ \frac{1}{2} m \dot{q}^2(t') + 2 \hbar \alpha [S_0(t') - S_0(t' - 2 L/c)] - \frac{\hbar^2}{4 m a^2(t')} \}] , \quad (2.7.5.7)$$

where $q(t)$ and $a(t)$ are solutions of the (2.7.4.10,11) differential equations.

Finally, it is important to note that putting $\alpha = 0$ into the eqs. (2.7.4.10,11) and (2.7.5.7) we obtain the free particle Feynman propagator. [2]

2.8. The Gross-Pitaevskii Equation

In 1961, Eugene P. Gross [36] and Lev Petrovich Pitaevskii [37] proposed a non-linear Schrödinger equation to represent time dependent physical systems, given by:

$$i\hbar \frac{\partial \Psi(x, t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x, t)}{\partial x^2} +$$

$$+ \{ V(x, t) + g \,|\, \Psi(x, t) \,|^2 \} \times \Psi(x, t) \,, \quad (2.8.1)$$

where $\Psi(x, t)$ and $V(x, t)$ are, respectively, the wavefunction and the time dependent potential of the physical system in study, and g is a constant.

2.8.1. The Wave Function of the Gross-Pitaevskii Equation

Putting the eqs. (2.1.1.1) and (2.1.1.2a,b) into the eq. (2.8.1), we have: [4,15.2,3]

$$i\hbar \left(\frac{\partial \varphi}{\partial t} + i\,\varphi \frac{\partial S}{\partial t} \right) = -\frac{\hbar^2}{2m} \left[\frac{\partial^2 \varphi}{\partial x^2} + 2\,i \frac{\partial S}{\partial x} \frac{\partial \varphi}{\partial x} + i\,\varphi \frac{\partial^2 S}{\partial x^2} - \varphi \left(\frac{\partial S}{\partial x} \right)^2 \right] +$$

$$+ [\, V(x, t) + g\, \varphi^2(x, t) \,] \times \varphi(x, t) \,, \quad (2.8.1.1)$$

Separating the real and imaginary parts of the eq. (2.8.1.1), results:

a) <u>imaginary part</u>

$$\frac{\partial \varphi}{\partial t} = -\frac{\hbar}{2m} \left(\varphi \frac{\partial^2 S}{\partial x^2} + 2 \frac{\partial \varphi}{\partial x} \frac{\partial S}{\partial x} \right) , \quad (2.8.1.2)$$

b) <u>real part</u>

$$-\hbar \frac{\partial S}{\partial t} = -\frac{\hbar^2}{2m} \left[\frac{1}{\varphi} \frac{\partial^2 \varphi}{\partial x^2} - \left(\frac{\partial S}{\partial x} \right)^2 \right] + V(x, t) + g\, \varphi^2 \,. \quad (2.8.1.3)$$

2.8.2. Dynamics of the Gross-Pitaevskii Equation

Now, let us see the correlation between the expressions (2.8.1.2-3) and the traditional equations of the Fluid Dynamics: [8] a) *Continuity Equation*, b) *Euler equation* (for conservative systems) or b') *Navier-Stokes equation* (for non-conservative systems). Thus, putting the eqs. (2.1.2.1a-f) into the eq. (2.8.1.2), and using the same operational protocol of the item 2.7.2, we obtain: [4,15.2,3]

$$\frac{\partial \varrho}{\partial t} + \frac{\partial (\varrho v_{qu})}{\partial x} = 0 \,, \quad (2.8.2.1)$$

expression that indicates <u>coherence</u> of the considered physical system represented by the Gross-Pitaevskii Equation (*GP−E*) [eq. (2.8.1)]; then the *Continuity Equation* it is preserved.

Now, let us obtained another dynamic equation of the *GP−E*. Considering the eqs. (2.1.2.1a-f) and (2.8.1.3), we have: [15.2,3]

$$\hbar \frac{\partial S(x, t)}{\partial t} + \frac{1}{2} m\, v_{qu}^2 + V(x, t) + V_{qu}(x, t) + V_{GP}(x, t) = 0 \,. \quad (2.8.2.2a)$$

where:

$$V_{GP}(x, t) = g\, \varphi^2(x, t) = g\, \varrho(x, t) \,, \quad (2.8.2.2b,c)$$

is the *Gross-Pitaevskii potential*.

Now, differentiating the eq. (2.8.1.3) with respect *x*, and using the eqs. (2.1.2.1a-f) and (2.8.2.2b,c), we have: [15.2,3]

$$\frac{\partial v_{qu}}{\partial t} + v_{qu} \frac{\partial v_{qu}}{\partial x} = - \frac{1}{m} \frac{\partial}{\partial x} (V + V_{qu} + V_{GP}) . \quad (2.8.2.3)$$

We observe that the eq. (2.8.2.3) has the aspect of the *Euler Equation* [8] for a fluid in movement.

Considering the *substantive differentiation* (local plus convective) or *hydrodynamic differentiation*, given by the eqs. (2.1.2.5a,b) and inserting into eq. (2.8.2.3), results:

$$m \frac{d^2 x}{dt^2} = - \frac{\partial}{\partial x} [\ V(x, t) + V_{qu}(x, t) + V_{GP}(x, t)\ ], \quad (2.8.2.4)$$

what has a form of the *Second Newton Law*, being the terms of the second member, respectively, the *classical newtonian force*, the *quantum bohmnian force* and the *Gross-Pitaevskii force*.

2.8.3 The Quantum Wave Packet of the Linearized Gross-Pitaevskii Equation along a Classical Trajectory

In order to find the quantum wave packet of the linearized Gross-Pitaevskii Equation (*GP−E*) along a classical trajectory, we must integrated the eq. (2.8.2.1). So, using the same protocol of the item 2.7.3, we have: [15.2,3]

$$v_{qu}(x, t) = \frac{\dot{a}(t)}{a(t)} [x - q(t)] + \dot{q}(t) . \quad (2.8.3.1)$$

We observe that the integration of the eq. (2.8.3.1) given the *bohmnian quantum trajectory* of the physical system represented by *GP−E*.

To obtain the quantum wave packet $\Psi(x , t)$ of the *GP−E* given by eq. (2.8.1), let us expand the functions S(x, t), V(xt), V_{qu}(x, t) and V_{GP}(x, t) around of $q(t) = <x>$ up to *second Taylor order*. In this way, using the eq. (2.1.3.14) we have:

$$S(x, t) = S[q(t), t] + S'[q(t), t] \times [x - q(t)] + \frac{1}{2} S''[q(t)\ t] \times [x - q(t)]^2, \quad (2.8.3.2)$$

$$V(x, t) = V[q(t), t] + V'[q(t), t] \times [x - q(t)] +$$

$$+ \frac{1}{2} V''[q(t), t] \times [x - q(t)]^2, \quad (2.8.3.3)$$

$$V_{qu}(x, t) = V_{qu}[q(t), t] + V'_{qu}[q(t), t] \times [x - q(t)] +$$

$$+ \frac{1}{2} V''_{qu}[q(t), t] \times [x - q(t)]^2, \quad (2.8.3.4a)$$

$$V_{qu}(x, t) = \frac{\hbar^2}{4\, m\, a^2(t)} - \frac{\hbar^2}{8\, m\, a^4(t)} \times [x - q(t)]^2 , \quad (2.8.3.4b)$$

$$V_{GP}(x, t) = V_{GP}[q(t), t] + V'_{GP}[q(t), t] \times [x - q(t)] +$$

$$+ \frac{1}{2} V''_{GP}[q(t), t] \times [x - q(t)]^2, \quad (2.8.3.5)$$

where (') and (") means, respectively, $\partial/\partial q$ and $\partial^2/\partial q^2$.

Expanding the eq. (2.1.3.1a) around of $q(t) = <x>$ up to *second Taylor order*, results:

$$\varrho(x, t) = \varrho[q(t), t] + \varrho'[q(t), t] \times [x - q(t)] + \frac{1}{2}\varrho''[q(t)\ t] \times [x - q(t)]^2 \ \rightarrow$$

$$\varrho(x, t) = [2\,\pi\,a^2(t)]^{-1/2} \times exp\,\{-\frac{[x - q(t)]^2}{2\,a^2(t)}\} +$$

$$+ [2\,\pi\,a^2(t)]^{-1/2} \times exp\,\{-\frac{[x - q(t)]^2}{2\,a^2(t)}\} \times \frac{1}{a^2(t)} \times [x - q(t)]^2 +$$

$$+ [2\,\pi\,a^2(t)]^{-1/2} \times exp\,\{-\frac{[x - q(t)]^2}{2\,a^2(t)}\} \times$$

$$\times \{-\frac{1}{2\,a^2(t)} + \frac{[x - q(t)]}{2\,a^4(t)}\} \times [x - q(t)]^2 . \qquad (2.8.3.6)$$

Using the expansion: $exp(-z) \simeq 1 - z$ in the eq. (2.8.3.6) and considering the terms up the second orders, we obtain:

$$\varrho(x, t) = [2\,\pi\,a^2(t)]^{-1/2} \times exp\,\{-\frac{[x - q(t)]^2}{2\,a^2(t)}\} \times$$

$$\times \{1 + \frac{[x - q(t)]^2}{a^2(t)} - \frac{[x - q(t)]^2}{2\,a^2(t)}\} =$$

$$= [2\,\pi\,a^2(t)]^{-1/2} \times \{1 - \frac{[x - q(t)]^2}{2\,a^2(t)}\} \times \{1 + \frac{[x - q(t)]^2}{2\,a^2(t)}\} =$$

$$= [2\,\pi\,a^2(t)]^{-1/2} \times \{1 + \frac{[x - q(t)]^2}{2\,a^2(t)} - \frac{[x - q(t)]^2}{2\,a^2(t)}\} \ \rightarrow$$

$$\varrho(x, t) = [2\,\pi\,a^2(t)]^{-1/2} . \qquad (2.8.3.7)$$

Inserting the eq. (2.8.3.7) in the eq. (2.8.2.2c), we have:

$$V_{GP}(x, t) = g\,[2\,\pi\,a^2(t)]^{-1/2} . \qquad (2.8.3.8)$$

Comparing the eqs. (2.8.3.5,8), results:

$$V_{GP}[q(t), t] = \frac{g}{a(t)\,\sqrt{2\,\pi}}\ ; \quad V_{GP}'[q(t), t] = 0\ ;\ V_{GP}''[q(t), t] = 0 . \qquad (2.8.3.9a\text{-}c)$$

Differenting the eq. (2.8.3.2) in the variable x, multiplying the result by $\hbar/m$ and using the eqs. (2.1.2.1c,d) and (2.8.3.1), we obtain:

$$\frac{\hbar}{m}\,\frac{\partial S(x, t)}{\partial x} = \frac{\hbar}{m}\,\{S'[q(t), t] + S''[q(t), t] \times [x - q(t)]\} = v_{qu}(x, t) =$$

$$= [\, \frac{\dot{a}(t)}{a(t)} \,] \times [x - q(t)] + \dot{q}(t) \;\; \rightarrow$$

$$S'[q(t), t] = \frac{m\,\dot{q}(t)}{\hbar}\,, \quad S''[q(t), t] = \frac{m}{\hbar}\,\frac{\dot{a}(t)}{a(t)}\,. \qquad (2.8.3.10a,b)$$

Substituting the eqs. (2.8.3.10a,b) into eq. (2.8.3.2), we have:

$$S(x, t) = S_0(t) + \frac{m\,\dot{q}(t)}{\hbar} \times [x - q(t)] + \frac{m}{2\,\hbar}\,\frac{\dot{a}(t)}{a(t)} \times [x - q(t)]^2\,, \qquad (2.8.3.11a)$$

where [see eq. (2.1.4.11b)]:

$$S_o(t) \equiv S[q(t), t]\,. \qquad (2.8.3.11b)$$

Differenting the eq. (2.8.3.11a) in relation to the time t and using the eq. (2.1.3.2), results (remember that $\partial x/\partial t = 0$): [15.2,3]

$$\frac{\partial S(x, t)}{\partial t} = \dot{S}_o(t) - \frac{m\,\dot{q}^2(t)}{\hbar} + \frac{m}{\hbar}\,\{\, \ddot{q}(t) - \dot{q}(t)\,[\, \frac{\dot{a}(t)}{a(t)} \,]\,\} \times [x - q(t)] +$$

$$+ \frac{m}{2\,\hbar}\,[\, \frac{\ddot{a}(t)}{a(t)} - \frac{\dot{a}^2(t)}{a^2(t)} \,] \times [x - q(t)]^2\,. \qquad (2.8.3.12)$$

Using the eqs. (2.1.2.1c-e) and (2.8.2.3), we obtain:

$$-\,\hbar\,\frac{\partial S}{\partial t} = [\, -\,\frac{\hbar^2}{2\,m}\,\frac{1}{\varphi}\,\frac{\partial^2 \varphi}{\partial x^2} + \frac{m}{2}\,(\, \frac{\partial S}{\partial x} \,)^2 \,] + V(x\,, t) + V_{GP}(x, t) \;\; \rightarrow$$

$$\hbar\,\frac{\partial S}{\partial t} + \frac{1}{2}\,m\,v_{qu}^2 + V(x, t) + V_{qu}(x, t) + V_{GP}(x, t) = 0\,. \qquad (2.8.3.13)$$

Inserting the eqs. (2.1.3.1b) and (2.8.3.1,3,5,9a-c,12) into eq. (2.8.3.13), ordering the result in potencies of $[x - q(t)]$, and considering that $[x - q(t)]^0 = 1$, we have: [15.2,3]

$$\{\hbar\,\dot{S}_o(t) - \frac{m}{2}\,\dot{q}^2(t) + V[q(t), t] + \frac{g}{a(t)\,\sqrt{2\,\pi}} + \frac{\hbar^2}{4\,m\,a^2(t)}\,\} \times [x - q(t)]^0 +$$

$$+ \{m\,\ddot{q}(t) + V'[q(t), t]\} \times [x - q(t)] +$$

$$+ \{\, \frac{m}{2}\,\frac{\ddot{a}(t)}{a(t)} + \frac{1}{2}\,V''[q(t), t] - \frac{\hbar^2}{8\,m\,a^4(t)}\,\} \times [x - (q)t)]^2 = 0\,. \qquad (2.8.3.14)$$

As the above relation [eq. (2.8.3.14)] is an identically null polynomium, the coefficients of the potencies must be all equal to zero, that is:

$$\dot{S}_o(t) = \frac{1}{\hbar}\,\{\, \frac{m}{2}\,\dot{q}^2(t) - V[q(t), t] - \frac{g}{a(t)\,\sqrt{\pi}} - \frac{\hbar^2}{4\,m\,a^2(t)}\,\}\,, \qquad (2.8.3.15)$$

$$\ddot{q}(t) + \frac{1}{m}\,V'[q(t), t] = 0\,, \qquad (2.8.3.16)$$

$$\ddot{a}(t) + \frac{1}{m} V''[q(t), t] \times a(t) = \frac{\hbar^2}{4\, m^2\, a^3(t)} . \quad (2.8.3.17)$$

Assuming that the following initial conditions are obeyed [see eqs. (2.1.3.21a-d):

$$q(0) = x_o , \quad \dot{q}(0) = v_o , \quad a(0) = a_o , \quad \dot{a}(0) = b_o , \quad (2.8.3.18a\text{-}d)$$

and that [see eqs. (2.1.2.1c,d) and (2.8.3.11b)]:

$$S_o(0) = \frac{m\, v_o\, x_o}{\hbar} , \quad (2.8.3.19)$$

the integration of the expression (2.8.3.15) will be given by:

$$S_o(t) = \frac{1}{\hbar} \int_o^t dt' \left\{ \frac{m}{2} \dot{q}^2(t') - V[q(t'), t'] - \frac{g}{a(t') \sqrt{2\,\pi}} - \frac{\hbar^2}{4\, m\, a^2(t')} \right\} +$$

$$+ \frac{m\, v_o\, x_o}{\hbar} . \quad (2.8.3.20)$$

Taking into account the eq. (2.8.3.20) in the eq. (2.8.3.11a) and considering the eq. (2.8.3.11b), results:

$$S(x, t) = \frac{1}{\hbar} \int_o^t dt' \left\{ \frac{m}{2} \dot{q}^2(t') - V[q(t'), t'] - \frac{g}{a(t') \sqrt{2\,\pi}} - \frac{\hbar^2}{4\, m\, a^2(t')} \right\} + \frac{m\, v_o\, x_o}{\hbar} +$$

$$+ \frac{m\, \dot{q}(t)}{\hbar} \times [x - q(t)] + \frac{m}{2\,\hbar} \frac{\dot{a}(t)}{a(t)} \times [x - q(t)]^2 . \quad (2.8.3.21)$$

The eq. (2.8.3.21) permit us, finally, to obtain the wave packet for the *GP−E*. Indeed, considering the eqs. (2.1.1.1) and (2.1.3.1b), we get: [15.2,3]

$$\Psi(x, t) = [2\,\pi\, a^2(t)]^{-1/4} \times exp \left\{ \left[\frac{i\, m}{2\,\hbar} \frac{\dot{a}(t)}{a(t)} - \frac{1}{4\, a^2(t)} \right] \times [x - q(t)]^2 \right\} \times$$

$$\times\, exp \left\{ \frac{i\, m\, \dot{q}(t)}{\hbar} \times [x - q(t)] + \frac{i\, m\, v_o\, x_o}{\hbar} \right\} \times$$

$$\times\, exp \left(\frac{i}{\hbar} \int_o^t dt' \left\{ \frac{m}{2}\, m\, \dot{q}^2(t') - \frac{g}{a(t') \sqrt{2\,\pi}} - V[q(t'), t'] - \frac{\hbar^2}{4\, m\, a^2(t')} \right\} \right) . \quad (2.8.3.22)$$

2.8.4. Calculation of the Feynman Propagator of the Linearized Gross-Pitaevskii Equation along a Classical Trajectory

For to calculated the Feynman-de Broglie-Bohm propagator of the linearized *GP−E* along a classical trajectory, will be following the same operational protocol of the item 2.7.4. Thus, by using the eqs. (2.1.4.1-3,5,9), we can write that: [15.2,3]

$$\Psi(x, t) = \int_{-\infty}^{+\infty} K(x, x_o; t, 0)\, \Psi(x_o, 0)\, dx_o , \quad (2.8.4.1)$$

$$\Phi(v_o, x, t) = (2\,\pi\, a_o^2)^{1/4}\, \Psi(v_o, x, t) , \quad (2.8.4.2)$$

$$\int_{-\infty}^{+\infty} dv_o\, \Phi^*(v_o, x', t)\, \Phi(v_o, x, t) = \left(\frac{2\pi\hbar}{m}\right) \delta(x - x')\,, \quad (2.8.4.3)$$

$$\varrho(v_o, x, t) = (2\pi a_o^2)^{1/4}\, \Phi^*(v_o, x, t)\, \Psi(v_o, x, t)\,, \quad (2.8.4.4)$$

$$\Psi(x, t) = \int_{-\infty}^{+\infty} \left\{ \left(\frac{m}{2\pi\hbar}\right) \int_{-\infty}^{+\infty} dv_o\, \Phi(v_o, x, t)\, \Phi^*(v_o, x_o, 0) \right\} \times$$

$$\times\, \Psi(x_o, 0)\, dx_o\,. \quad (2.8.4.5)$$

Comparing the eqs. (2.8.4.1,5), we have:

$$K(x, x_o, t) = \left(\frac{m}{2\pi\hbar}\right) \int_{-\infty}^{+\infty} dv_0\, \Phi(v_o, x, t)\, \Phi^*(v_o, x_o, 0)\,. \quad (2.8.4.6)$$

Substituting the eqs.(2.8.4.1,2) in the eq. (2.8.4.6), we obtain the Feynman Propagator of the linearized *GP−E* along a classical trajectory, that we were looking for, that is [remembering that $\Phi^*(v_o, x_o, 0) = exp\,(-\frac{i\, m\, v_o\, x_o}{\hbar})$]:

$$K(x, x_o; t) = \left(\frac{m}{2\pi\hbar}\right) \int_{-\infty}^{+\infty} dv_o \sqrt{\frac{a_0}{a(t)}} \times$$

$$\times\, exp\left[\left\{ \frac{i}{2}\, \frac{m}{\hbar}\, \frac{\dot{a}(t)}{a(t)} - \frac{1}{4\, a^2(t)} \right\} \times [x - q(t)]^2 \right] \times$$

$$\times\, exp\left\{ \frac{i\, m\, \dot{q}(t)}{\hbar} \times [x - q(t)] \right\} \times$$

$$\times\, exp\left(\frac{i}{\hbar} \int_o^t dt' \left\{ \frac{m}{2}\dot{q}^2(t') - \frac{g}{a(t')\sqrt{2\pi}} - V[q(t'), t'] - \frac{\hbar^2}{4\, m\, a^2(t')} \right\} \right), \quad (2.8.4.7)$$

where $q(t)$ and $a(t)$ are solutions of the differential equations given by the eqs.(2.8.3.16,17).

Finally, it is important to note that putting $g = 0$ and $V[q(t'), t'] = 0$ into eqs. (2.8.3.16,17) and (2.8.4.7), we obtain the free Feynman propagator. [2]

3. Conclusions

In this article, we calculated the *Feynman Propagator* for eight non-linear Schrödinger equations. To obtain these propagators we adopted the quantum mechanical formalism of the de Broglie-Bohm. This was done because this formalism permits to perform essential linear approximations along the classical trajectories that are the basic ingredients of the of the Feynman's principle of minimum action of quantum mechanics. Besides, in particular case of the free particle, our results confirms the calculations realized by Feynman and Hibbs. [2]

3. Conclusions

In this article, we calculated the *Feynman Propagator* for many non-linear Schrödinger equations. To obtain these propagators we adopted the quantum mechanical formalism of the de Broglie-Bohm. This was done because this formalism permits to perform essential linear approximations along the classical trajectories that are the basic ingredients of the of the Feynman's principle of minimum action of quantum mechanics. Besides, in particular case of the free particle, our results confirms the calculations realized by Feynman and Hibbs. [2]

NOTES AND REFERENCES

1. FEYNMAN, R. P. 1948. *Reviews of Modern Physics* **20**: 367.

2. FEYNMAN, R. P. and HIBBS, A. R. 1965. **Quantum Mechanics and Path Integrals**, McGraw-Hill Book Company.

3. BERNSTEIN, I. B. 1985. *Physical Review* **A32**: 1.

4. BASSALO, J. M. F., ALENCAR, P. T. S., CATTANI, M. S. D. e NASSAR, A. B. 2002. **Tópicos da Mecânica Quântica de de Broglie-Bohm**, EDUFPA; -----. 2010. E-Book (http://publica-sbi.if.usp.br/PDFs/pd1655.pdf.

5. BIALYNICKI-BIRULA, I. and MYCIELSKI, J. 1976. *Annals of Physics (N.Y)* **100**: 62; -----. 1979. *Physica Scripta* **20**: 539.

6. MADELUNG, E. 1926. *Zeitschrift für Physik* **40**: 322.

7. BOHM, D. 1952. *Physical Review* **85**: 166.

8. See books on the Fluid Mechanics, for instance:

8.1. STREETER, V. L. and DEBLER, W. R. 1966. **Fluid Mechanics**, McGraw-Hill Book Company, Incorporation.

8.2. COIMBRA, A. L. 1967. **Mecânica dos Meios Contínuos**, Ao Livro Técnico S. A.

8.3. LANDAU, L. et LIFSHITZ, E. 1969. **Mécanique des Fluides**. Éditions Mir.

8.4. BASSALO, J. M. F. 1973. **Introdução à Mecânica dos Meios Contínuos**, EDUFPA.

8.5. CATTANI, M. S. D. 1990/2005. **Elementos de Mecânica dos Fluidos**, Edgard Blücher.

9. NASSAR, A. B., BASSALO, J. M. F., ALENCAR, P. T. S., CANCELA, L. S. G. and CATTANI, M. S. D. 1977. *Physical Review* **E56**: 1230.

10. BATEMAN, H. 1931. *Physical Review* **38**: 815.

11. CALDIROLA, P. 1941. *Nuovo Cimento* **18**: 393.

12. KANAI, E. 1948. *Progress in Theoretical Physics* **3**: 440.

13. DIÓSI, L. and HALLIWELL, J. J. 1998. *Physical Review Letters* **81**: 2846.

14. NASSAR, A. B. 2004. **Chaotic Behavior of a Wave Packet under Continuous Quantum Mechanics** (*preprint* DFUFPA).

15. See articles of the authors, in *arXiv.org*:

15.1 BASSALO, J. M. F., ALENCAR, P. T. S., SILVA, D. G. da, NASSAR, A. B. and CATTANI, M. S. D. 2009a. *arXiv:0905.4280v1*[quant-ph], 26 May.

15.2 BASSALO, J. M. F., ALENCAR, P. T. S., SILVA, D. G. da, NASSAR, A. B. and CATTANI, M. S. D. 2009b. *arXiv:0910.5160v1*[quant-ph], 27 October.

15.3 BASSALO, J. M. F., ALENCAR, P. T. S., SILVA, D. G. da, NASSAR, A. B. and CATTANI, M. S. D. 2010a. *arXiv:1001.3384v1*[quant-ph], 19 January.

15.4 BASSALO, J. M. F., ALENCAR, P. T. S., SILVA, D. G. da, NASSAR, A. B. and CATTANI, M. S. D. 2010b. *arXiv:1004.1416v1*[quant-ph], 08 April.

15.5 BASSALO, J. M. F., ALENCAR, P. T. S., SILVA, D. G. da, NASSAR, A. B. and CATTANI, M. S. D. 2010c. *arXiv:1006.1868v1*[quant-ph], 09 June.

15.6 BASSALO, J. M. F., ALENCAR, P. T. S., SILVA, D. G. da, NASSAR, A. B. and CATTANI, M. S. D. 2010d. *arXiv:1010.2640v1*[quant-ph], 13 October.

15.7 BASSALO, J. M. F., ALENCAR, P. T. S., SILVA, D. G. da, NASSAR, A. B. and CATTANI, M. S. D. 2011. *arXiv:1101.0688v1*[quant-ph], 04 January.

16. KOSTIN, M. D. 1972. *Journal of Chemical Physics* **57**: 3539.

17. SCHUCH, D., CHUNG, K. M. and HARTMANN, H. 1983. *Journal of Mathematical Physics* **24**: 1652; -----. 1984. *Journal of Mathematical Physics* **25**: 3086; -----. 1985. *Berichte der Bunsen-Gesellschaft für Physikalische Chemie* **89**: 589.

18. NASSAR, A. B. 1986. *Journal of Mathematical Physics* **27**: 2949.

19. GRADSHTEYN, I. S. and RYZHIK, I. W. 1965. **Table of Integrals, Series and Products**, Academic Press.

20. SÜSSMANN, D. 1973. *Seminar Talk at Los Alamos*.

21. HASSE, R. W. 1975. *Journal of Mathematical Physics* **16**: 2005.

22. ALBRECHT, K. 1975. *Physics Letters* **B56**: 127.

23. KOSTIN, M. D. 1975. *Journal of Statistics Physics* **12**: 146.

24. LORENTZ, H. A. 1892. *Archives Neérlandaises des Sciences Exactes et Naturales 25*, 363552; -----. 1909. **Theory of Electrons**, Leipzig, Teubner.

25. ABRAHAM, A. 1905. **Theorie der Elektrizitat II**, Leipzig, Teubner.

26. ROHRLICH, F. 1965. **Classical Charged Particles**, Addison-Wesley; ----- 1997/2000. *American Journal of Physics* **65**: 1051; **68**: 1109..

27. GRIFFITHS, D. 2004. **Introductiuon to Electrodynamics**, Prentice Hall).

28. SOMMERFELD, A. 1904. *Akademie van Wetensch te Amsterdam* **13**.

29. PAGE, L. 1918. *Physical Review* **11**: 376.

30. KIM, K. J. and SESSLER, A. M. 1998. *8th Workshopo on Advanced Acceleration Concepts*, July 5-11.

31. JACKSON, J. D. 1998. **Classical Electrodynamics**, John Wiley.

32. MONIZ, E. J. and SHARP, D. H. 1977. *Physical Review* **D15**: 2850.

33. LEVINE, H., MONIZ, E. J. and SHARP, D. H. 1977. *American Journal of Physics* **45**: 75.

34. COOK, R. J. 1984. *American Journal of Physics* **52**: 894.

35. NASSAR, A. B. 2007. *International Journal of Theoretical Physics* **46**: 548.

36. GROSS, E. P. 1961. *Nuovo Cimento* **20**: 1766.

37. PITAEVSKII, L. P. 1961. *Soviet Physics (JETP)* **13**: 451.